

On Trace Averages and Sharp Asymptotics for Powers of Haar-Unitary Matrices

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For a Haar-distributed matrix $U \in \mathbf{U}(N)$, Rains's power-map theorem represents the spectrum of U^m , $m \in \mathbb{N}$, as a superposition of independent lower-dimensional CUE spectra. We determine how this exact decomposition appears in trace moments and in sharp Gaussian and Poisson limit laws. For arbitrary mixed trace products, a Charlier–Ewens convolution separates the stable Wick term from its full finite- N correction, while a displacement determinant gives a direct evaluation. A compatible Fredholm factorisation yields sharp total-variation asymptotics for the normalised trace in four regimes, from fixed powers to $m \geq N$. For a fixed trace power p , coherently summing the first correction band yields the exact factor $e^{2/p}$. At the mean-spacing scale, we obtain a sharp Poisson limit for the spectrum of U^m ; its first-order correction, obtained via the second Charlier polynomial, quantifies the residual under-dispersion of local eigenvalue counts.

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1 Introduction

1.1 Power maps and the emergence of independence

Let U_N be an $N \times N$ Haar-distributed unitary matrix; it has eigenvalues $e^{i\theta_1}, \dots, e^{i\theta_N}$ that lie on the unit circle. Each eigenvalue has a uniform marginal distribution, but jointly the eigenvalues exhibit strong repulsion, which produces a configuration far more evenly spaced than independent uniform points. The power map

$$U_N \mapsto U_N^m$$

acts on the eigenangles by $\theta \mapsto m\theta \pmod{2\pi}$ with $m \in \mathbb{N}$. It folds the circle over itself, so that several distinct points share a common image. After folding, one might expect a complicated rearrangement of CUE correlations. Instead, Rains showed that the spectrum decomposes cleanly into independent unitary spectra [6]. In particular, once $m \geq N$, the eigenvalues of U_N^m are distributed exactly as N independent uniform points on the unit circle [5].

Let

$$N = mq + r, \quad 0 \leq r < m, \quad (1)$$

be the Euclidean division of N by m . Rains proved the distributional identity

$$\text{spec}(U_N^m) \stackrel{d}{=} \bigsqcup_{a=1}^r \text{spec}(V_{q+1,a}) \sqcup \bigsqcup_{a=r+1}^m \text{spec}(V_{q,a}), \quad (2)$$

where the matrices on the right-hand side are mutually independent and Haar distributed in the indicated unitary groups. Thus, in law, the spectrum of U_N^m is the superposition of r independent CUE spectra of size $q + 1$ and $(m - r)$ of size q . After zero-dimensional blocks are discarded, there are $\min(m, N)$ such blocks, each containing at most $\lceil N/m \rceil$ eigenvalues.

The decomposition identifies exactly which correlations the power map removes. Eigenvalues belonging to different blocks are independent, whereas those within the same block retain the characteristic repulsion of a CUE spectrum. Increasing m therefore trades a few large blocks for many smaller, independent ones. When $m = o(N)$, the blocks remain large and much of the local CUE structure survives. When $m \asymp N$, their dimensions remain bounded while their number grows proportionally to N . At the endpoint $m \geq N$, precisely N one-dimensional blocks remain and exact independence follows.

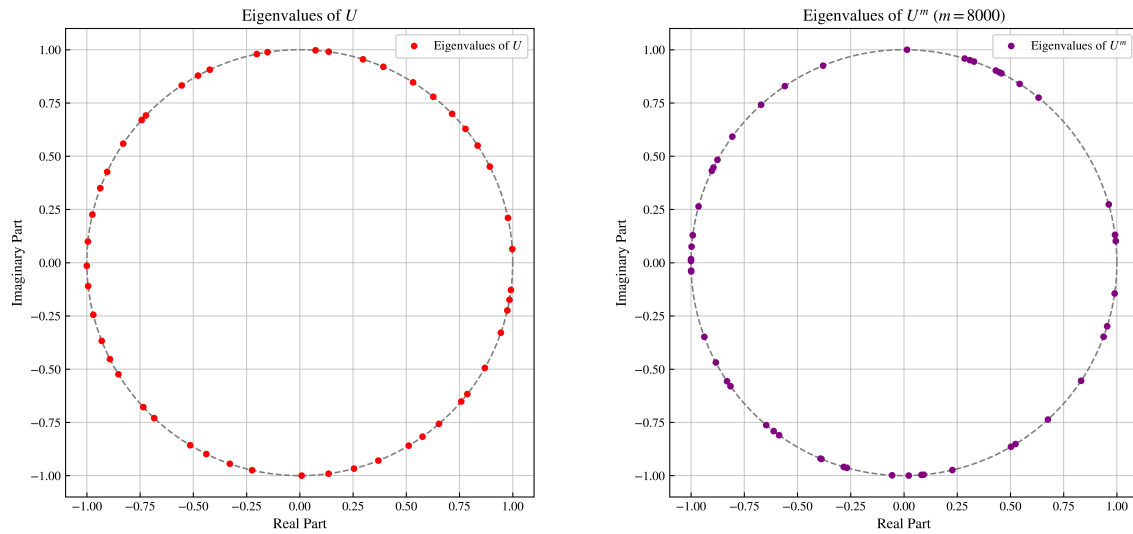


Figure 1: A comparison of the eigenvalue distribution for U and U^m with $m \geq N$ and $U \in U(N)$.

1.2 Trace moments: exact identities and finite- N corrections

To convert this spectral picture into a global scalar statistic, consider the empirical spectral measure and its Fourier transform. For eigenvalues $e^{i\theta_1}, \dots, e^{i\theta_N}$, set

$$\mu_{U_N} := \frac{1}{N} \sum_{a=1}^N \delta_{\theta_a}, \quad \hat{\mu}_{U_N}(m) = \frac{1}{N} \sum_{a=1}^N e^{im\theta_a} = \frac{1}{N} \text{Tr}(U_N^m). \quad (3)$$

Thus $N^{-1} \text{Tr}(U_N^m)$ is the m th Fourier coefficient of the spectrum; it averages the unit complex numbers $e^{im\theta_a}$ and its magnitude measures the cancellation in their complex-vector sum. The trace in (3) belongs to a classical theory of exact CUE averages. In 1962, Dyson's analysis of the CUE kernel [3] gave the covariance identity

$$\mathbb{E} \left[\text{Tr}(U_N^j) \overline{\text{Tr}(U_N^k)} \right] = \mathbf{1}_{\{j=k\}}(j \wedge N), \quad j, k \geq 1, \quad (4)$$

where $a \wedge b := \min(a, b)$.

Later, representation theory provided a natural language for arbitrary mixed trace averages. A product of traces is a power-sum symmetric function of the eigenvalues, and Frobenius's formula decomposes it into Schur characters [4]. Since these characters are orthonormal for Haar measure on $U(N)$, the unitary integral can be evaluated exactly as a finite character sum. Explicitly, let $\mathbf{p} = (p_1, \dots, p_{k_1})$ and $\mathbf{q} = (q_1, \dots, q_{k_2})$ be positive integer vectors and let

$$\mu = (1^{a_1} 2^{a_2} \dots), \quad \nu = (1^{b_1} 2^{b_2} \dots) \quad (5)$$

be their exponent types; thus $a_j = \#\{i : p_i = j\}$ and $b_j = \#\{i : q_i = j\}$. Write

$$C_N(\mathbf{p}, \mathbf{q}) := \mathbb{E} \left[\prod_{i=1}^{k_1} \text{Tr}(U_N^{p_i}) \prod_{j=1}^{k_2} \overline{\text{Tr}(U_N^{q_j})} \right] \quad (6)$$

and note that multiplication of U_N by a scalar phase shows immediately that $C_N(\mathbf{p}, \mathbf{q}) = 0$ unless $\sum_{i=1}^{k_1} p_i = \sum_{j=1}^{k_2} q_j =: D$. Frobenius's character formula and Schur orthogonality yield the exact finite character sum

$$C_N(\mathbf{p}, \mathbf{q}) = \sum_{\substack{\lambda \vdash D \\ \ell(\lambda) \leq N}} \chi_\mu^\lambda \chi_\nu^\lambda. \quad (7)$$

Here χ_μ^λ is the irreducible character of the symmetric group S_D labelled by the partition λ , evaluated on any permutation of cycle type μ . A separate approach to Haar integration originates in Weingarten's 1978 formula which operates directly at the level of matrix entries [9]. It expresses integrals of products of unitary matrix entries and their complex conjugates as sums over permutations; this approach was later developed into a systematic integration calculus by Collins and Śniady [1], providing new expressions for (7). In 1994, Diaconis and Shahshahani then identified the decisive simplification of the character sum above [2]. If $\mathbf{a} = (a_j)_{j \geq 1}$ and $\mathbf{b} = (b_j)_{j \geq 1}$ are finitely supported and $\sum_{j \geq 1} j(a_j + b_j) \leq 2N$, then

$$C_N(\mathbf{p}, \mathbf{q}) = \mathbf{1}_{\{\mathbf{a}=\mathbf{b}\}} \prod_{j \geq 1} j^{a_j} a_j! \quad (8)$$

or, equivalently, every partition $\lambda \vdash D$ automatically satisfies $\ell(\lambda) \leq N$. The length restriction in (7) therefore discards no contributions, and the full character orthogonality relation gives $\sum_{\lambda \vdash D} \chi_\mu^\lambda \chi_\nu^\lambda = \mathbf{1}_{\{\mu=\nu\}} \prod_{j \geq 1} j^{a_j} a_j!$ recovering (8). We refer to $D \leq N$ as the stable range, i.e. the range in which the simplified form (8) holds; here, the non-vanishing condition $\mathbf{a} = \mathbf{b}$ implies $k_1 = k_2 := k$. Outside this

range, k_1 and k_2 may differ in contributing terms and $C_N(\mathbf{p}, \mathbf{q})$ takes a correspondingly more complicated form. The identity (8) has a natural Gaussian interpretation, which is made transparent by Wick's rule. Let $(G_j)_{j \geq 1}$ be independent centred complex Gaussians satisfying

$$\mathbb{E}[G_j \overline{G_k}] = j \mathbf{1}_{\{j=k\}}, \quad \mathbb{E}[G_j^2] = 0. \quad (9)$$

Then, by Wick's rule, one has

$$\mathbb{E} \left[\prod_{i=1}^k G_{p_i} \prod_{j=1}^k \overline{G_{q_j}} \right] = \sum_{\sigma \in S_k} \prod_{i=1}^k p_i \mathbf{1}_{\{p_i = q_{\sigma(i)}\}} = \mathbf{1}_{\{\mathbf{a}=\mathbf{b}\}} \prod_{j \geq 1} j^{a_j} a_j!, \quad (10)$$

and thus (8) states that $C_N(\mathbf{p}, \mathbf{q})$ agrees exactly with the corresponding Gaussian moment throughout the stable range. This range must not, however, be confused with exact Gaussianity of the traces themselves. For fixed N , we have $|\text{Tr}(U_N^m)| \leq N$, whereas a non-degenerate complex Gaussian has unbounded support. The assertion instead concerns a growing initial collection of moments. In particular, for fixed m ,

$$\mathbb{E} |\text{Tr}(U_N^m)|^{2k} = m^k k! \quad \text{whenever} \quad k \leq \left\lfloor \frac{N}{m} \right\rfloor. \quad (11)$$

Thus, the number of even moments that match their Gaussian counterparts grows on the scale N/m .

Cumulants provide a complementary description of the transition beyond the stable range. For non-zero integers $r_1, \dots, r_s$, let $\kappa_s^{(N)}(r_1, \dots, r_s)$ denote the joint cumulant of $\text{Tr}(U_N^{r_1}), \dots, \text{Tr}(U_N^{r_s})$, with $\text{Tr}(U_N^{-r}) := \overline{\text{Tr}(U_N^r)}$; as for the moments themselves, it vanishes unless $\sum_a r_a = 0$. Any cumulants of order at least three vanish identically for a Gaussian family, so their survival measures a departure from Gaussian behaviour. Soshnikov expressed these trace cumulants as an alternating sum over ordered set partitions [7]. When

$$\sum_a |r_a| \leq 2N,$$

every such higher cumulant vanishes, leaving only the covariance and hence Wick's rule. Beyond this range, the finite- N cutoff interrupts the cancellation and higher-order connected terms may survive; Soshnikov and Wu analysed these surviving terms for frequencies of order N [8]. Recovering a full mixed moment from the cumulants, however, still requires summing over set partitions via the moment-cumulant relation.

Other approaches capture trace averages through different structures. Pastur and Vasilchuk derived integration-by-parts recurrences for trace moments in the classical compact groups, while Adler and van Moerbeke placed multitime unitary integrals within the Toeplitz-lattice and reduced 2-Toda framework. Duits and Johansson used Toeplitz determinants to study powers varying with N , and Jiang and Matsumoto obtained related power-sum moment estimates for the circular β -ensembles [21, 27, 11, ?]. These methods yield recurrences, generating functions, or asymptotic formulas, rather than a direct separation of the finite- N moment from its Gaussian part.

The character and cumulant formulas therefore encode the finite- N correction in different coordinates: the former imposes the length cutoff inside a partition sum, while the latter first extracts the connected contributions. We instead separate the Wick term at the level of the mixed moment itself:

$$C_N(\mathbf{p}, \mathbf{q}) = \mathbf{1}_{\{k_1=k_2\}} \sum_{\pi \in S_{k_1}} \prod_{i=1}^{k_1} p_i \mathbf{1}_{\{p_i = q_{\pi(i)}\}} + \mathcal{R}_N(\mathbf{p}, \mathbf{q}), \quad (12)$$

where $\mathcal{R}_N(\mathbf{p}, \mathbf{q}) = 0$ for $D \leq N$. Theorem 11 resolves $\mathcal{R}_N$ into finitely many Charlier–Ewens exterior bands, while Theorem 12 evaluates $C_N(\mathbf{p}, \mathbf{q})$ directly, by a finite determinant that bypasses the partition

sum entirely. Only bands satisfying

$$\ell(N + \ell) \leq D \quad (13)$$

can contribute, which identifies both the onset and the finite depth of the correction.

1.3 Global trace fluctuations and total variation

As Haar invariance gives $\mathbb{E} \operatorname{Tr}(U_N^m) = 0$, (4) yields

$$X_{N,m} := \frac{\operatorname{Tr}(U_N^m)}{\sqrt{m \wedge N}}, \quad \mathbb{E}|X_{N,m}|^2 = 1. \quad (14)$$

This is the natural unit-variance normalisation of the m th trace.

A different exact perspective comes from Rains's power-map theorem [22, 23]. Although we used it above to motivate the spectral transition, it also has an immediate consequence for traces. Taking traces in (2) gives

$$X_{N,m} \stackrel{d}{=} \frac{1}{\sqrt{m \wedge N}} \left(\sum_{a=1}^r \operatorname{Tr}(V_{q+1,a}) + \sum_{a=r+1}^m \operatorname{Tr}(V_{q,a}) \right). \quad (15)$$

The normalised trace is therefore a sum of independent block traces. For fixed m , there are finitely many blocks whose dimensions grow; when $m \asymp N$, the block dimensions remain bounded but the number of independent summands grows. These are two distinct routes to the same Gaussian limit.

The first probability question of the paper is therefore: how accurately can the law of $X_{N,m}$ be replaced by a standard complex Gaussian? Let Z have density

$$g(x) = \pi^{-1} e^{-|x|^2}, \quad x \in \mathbb{C} \simeq \mathbb{R}^2. \quad (16)$$

We measure the error in total variation,

$$d_{\text{TV}}(\mu, \nu) := \sup_A |\mu(A) - \nu(A)| = \frac{1}{2} \int |f_\mu - f_\nu| dx, \quad (17)$$

when μ and ν have densities f_μ and f_ν . This distance controls the error for every measurable event, not merely moments or smooth tests, and is unchanged by measurable bijections. Determining its first asymptotic term is accordingly stronger than proving a central limit theorem or an upper bound.

Equation (15) reveals several routes to Gaussian behaviour. If $m = p$ is fixed, the first non-Gaussian moment is pushed to an order proportional to N , which creates a factorially small error. If $m \asymp N$ and $q = \lfloor N/m \rfloor$ is fixed, $X_{N,m}$ is a sum of increasingly many independent traces of bounded-size Haar matrices, so a classical triangular-array mechanism applies. If both m and q diverge, the two mechanisms coexist. Finally, for $m \geq N$ the summands are independent uniform phases. In every case $X_{N,m}$ tends to Z as $N \rightarrow \infty$, but the leading total-variation correction remembers which mechanism produced the Gaussian limit.

Two complementary traditions quantify this convergence. Stein and heat-kernel methods exploit differential identities on the compact group; they give explicit bounds for Gaussian approximation and extend naturally to vectors of trace powers [10, 13]. Toeplitz-determinant and Fourier methods retain more of the characteristic function, which makes much smaller errors visible: Johansson and Lambert obtained super-exponential multivariate approximation for low powers, and Courteaut, Johansson and Lambert produced bounds interpolating between the Berry–Esseen and super-exponential regimes [17, 5]. Meckes [20] gives a systematic account of normal approximation on the compact classical groups. In a parallel arithmetic setting, Gorodetsky and Rodgers established a function-field analogue of asymptotic trace independence [15].

The stable range should not be confused with exact Gaussianity. It says that all mixed moments below a moving degree threshold agree exactly with Gaussian moments. For fixed N , however, $\text{Tr}(U^p)$ is bounded whereas a Gaussian is not. If $N = pq + r$, the first defective moment in the sequence $\mathbb{E}|\text{Tr}(U_N^p)|^{2k}$ has index $k = q + 1$. Fourier inversion turns that delayed defect into a Laguerre mode of factorial size. Our fixed-power theorem determines the complete leading term:

$$d_{\text{TV}}\left(\frac{\text{Tr}(U_N^p)}{\sqrt{p}}, Z\right) \sim c_* \frac{(p-r)e^{2/p}}{p^{q+1}\Gamma(q+2)(q+1)^{1/4}}, \quad c_* := \frac{2^{3/4}\Gamma(3/4)}{\pi^{3/2}}. \quad (18)$$

The first altered moment gives the correct factorial scale, but not the leading constant: the factor $e^{2/p}$ is produced by coherent summation over the complete first correction band. The four global theorems in Section 2 give sharp asymptotics in the regimes above.

1.4 Local spectral statistics and Poisson approximation

Two related probability questions run through the calculation. The first concerns the global Fourier statistic $\text{Tr}(U^m)$ and asks how close it is to a complex Gaussian. The second unfolds the eigenvalues of U^m at mean spacing and asks whether the resulting point process is close to a Poisson process on a fixed interval. Both questions measure a loss of the correlation present in CUE, but their limiting mechanisms are different.

The distinct Gaussian mechanisms were described in Section 1.3. For the local problem, write the eigenvalues of U_N^m as $e^{i\theta_1}, \dots, e^{i\theta_N}$ and unfold them to mean spacing one:

$$\Xi_{N,m} := \sum_{j=1}^N \delta_{N\theta_j/(2\pi)} \quad \text{on } \mathbb{T}_N = \mathbb{R}/N\mathbb{Z}. \quad (19)$$

For a fixed interval B of length L and a unit-rate Poisson process Π_B on B , we study

$$d_{\text{TV}}(\mathcal{L}(\Xi_{N,m}|_B), \mathcal{L}(\Pi_B)). \quad (20)$$

After unfolding, each independent Rains block contributes only a small fraction of the unit intensity. A block is therefore primarily a zero-or-one source, and their superposition becomes Poisson even though the points within one block retain determinantal repulsion.

If $N = mq + r$, the block intensity fractions are $(q+1)/N$ for r blocks and q/N for the remaining $m-r$. Their quadratic mass is

$$\beta_{N,m} = \frac{r(q+1)^2 + (m-r)q^2}{N^2}. \quad (21)$$

Theorem 6 proves, uniformly whenever $N, m \rightarrow \infty$,

$$d_{\text{TV}}(\mathcal{L}(\Xi_{N,m}|_B), \mathcal{L}(\Pi_B)) \sim A(L)\beta_{N,m}, \quad A(L) := \frac{1}{4} \mathbb{E}|(Z_L - L)^2 - Z_L|, \quad (22)$$

where Z_L is Poisson with mean L . This second Poisson–Charlier mode records that a local eigenvalue count is slightly under-dispersed relative to a Poisson variable; positional repulsion within one block contributes only at the next order.

Barbour and Brown and Barbour, Holst and Janson develop total-variation approximation for Poisson variables and point processes through the Stein–Chen method [1, 2]; Hough, Krishnapur, Peres and Virág treat determinantal processes and their kernels in [16]. We are not aware of a previous sharp total-variation asymptotic, including its leading constant, for the local spectrum of U_N^m throughout the unrestricted regime $N, m \rightarrow \infty$.

Charlier polynomials consequently appear twice, for two different reasons. In the exact global moment formula they diagonalise the independent Poisson cycle counts underlying the Ewens measure on partitions. In the local problem they are the orthogonal correction modes of the Poisson distribution itself. The two appearances organise different limits, but both expose the first residual dependence after Rains’s block decomposition.

1.5 Brief note on results

The main contributions fall into three connected groups. The first concerns exact averages of traces beyond the stable range; the other two determine sharp probability distances at the global and local scales.

Problem	Principal representation	Consequence
Mixed trace moments	Charlier–Ewens convolution and displacement determinant	Exact evaluation for arbitrary mixed exponents and an explicit separation of the Wick term from $\mathcal{R}_N$ and its correction bands.
Global Gaussian law	Fredholm factorisation and Fourier–Laguerre transfer	Sharp total-variation asymptotics in four power regimes, including the coherent factor $e^{2/p}$ for fixed powers.
Local Poisson law	Rains blocks and a Poisson–Charlier expansion	A sharp process-level and count-level asymptotic, uniform whenever $N, m \rightarrow \infty$.

The foundations on which these results rest are classical. They include Rains’s spectral decomposition [22, 23], the stable mixed moments of Diaconis and Shahshahani [8], and Soshnikov’s cumulant formula [24]. The Toeplitz and symmetric-function arguments use Heine–Szegő, Gessel’s identity, and the Borodin–Okounkov–Geronimo–Case formula [14, 4, 3]. For the Fredholm estimates, we use Courteaut, Johansson and Lambert and state the parameter translation explicitly at each application [5]. The new content lies in the exact forms assembled from these inputs, the resulting description of the finite- N correction, and the sharp total-variation asymptotics obtained from it.

Section 2 contains the main probability theorems and the exact trace identities. Within it, Subsection 2.2 develops the Charlier–Ewens correction, the displacement determinant, and the multitime Fredholm factorisation. Section 3 collects the Fourier–Laguerre transfer principle and the Bessel Fredholm estimate used throughout the Gaussian approximation. The four global regimes are then treated in Sections 4–7. Section 8 proves the local Poisson theorem.

2 Main results

We identify $\mathbb{C}$ with $\mathbb{R}^2$ and retain the notation Z , g , and d_{TV} introduced in Section 1. Our asymptotic notation records the fixed parameters: constants in $O_L(\cdot)$ may depend on the interval length L , and constants in $O_q(\cdot)$ may depend on the fixed quotient q . In the Fourier arguments we repeatedly use

$$a = \frac{|t|^2}{4}, \quad \rho = \frac{|t|}{\sqrt{m}}, \quad \nu = q + 1, \quad (23)$$

where m denotes the trace power (and $m = p$ in the fixed-power regime).

Theorem 1 (Fixed powers). *Fix $p \in \mathbb{N}$. Let U_N be Haar distributed on $U(N)$ and write*

$$N = pq + r, \quad 0 \leq r < p, \quad \nu = q + 1. \quad (24)$$

Then, as $N \rightarrow \infty$,

$$d_{\text{TV}}\left(\frac{\text{Tr}(U_N^p)}{\sqrt{p}}, Z\right) = \frac{(p-r)e^{2/p}}{2p^\nu \Gamma(\nu+1)} \int_0^\infty e^{-u} |L_\nu(u)| du + o\left(\frac{1}{p^\nu \Gamma(\nu+1) \nu^{1/4}}\right) \quad (25)$$

$$\sim \frac{2^{3/4} \Gamma(3/4)}{\pi^{3/2}} \frac{(p-r)e^{2/p}}{p^\nu \Gamma(\nu+1) \nu^{1/4}}, \quad (26)$$

where $L_\nu = L_\nu^{(0)}$ is the ordinary Laguerre polynomial. The error terms are uniform in the finitely many residues $r \in \{0, \dots, p-1\}$.

Remark 2. The first altered moment occurs at order $\nu = q+1$, but it does not by itself give the leading total-variation constant. Every fixed shift $L_{\nu+j}$ in the first exterior band has the same leading oscillatory profile. Their coefficients converge to $(2/p)^j/j!$, and hence add up to $e^{2/p}$. This is the point at which the full correction formula contains more information than the location of the first defective moment.

Theorem 3 (Independent-phase regime). *Let U_N be Haar distributed on $U(N)$ and let $m_N \geq N$ be any integer. Then*

$$d_{\text{TV}}\left(\frac{\text{Tr}(U_N^{m_N})}{\sqrt{N}}, Z\right) = \frac{e^{-2}(\sqrt{2} \cosh(\sqrt{2}) - \sinh(\sqrt{2}))}{N} + O(N^{-2}). \quad (27)$$

The estimate is uniform over all choices of $m_N \geq N$.

Theorem 4 (Joint growth of power and block dimension). *Let $m = m_N < N$, write*

$$N = mq + r, \quad 0 \leq r < m, \quad \nu = q+1, \quad (28)$$

and suppose that $m \rightarrow \infty$ and $q \rightarrow \infty$. Then, uniformly in the remainder r ,

$$d_{\text{TV}}\left(\frac{\text{Tr}(U_N^m)}{\sqrt{m}}, Z\right) = \frac{m-r}{2m^\nu \Gamma(\nu+1)} \int_0^\infty e^{-u} |L_\nu(u)| du + o\left(\frac{m-r}{m^\nu \Gamma(\nu+1) \nu^{1/4}}\right) \quad (29)$$

$$\sim \frac{2^{3/4} \Gamma(3/4)}{\pi^{3/2}} \frac{m-r}{m^{q+1} \Gamma(q+2) (q+1)^{1/4}}. \quad (30)$$

No relation between the rates at which m and q diverge is required.

Theorem 5 (Fixed block dimension). *Fix $q \geq 1$. Let $m = m_N \rightarrow \infty$ and write*

$$N = mq + r, \quad 0 \leq r < m, \quad s = m - r, \quad \nu = q+1. \quad (31)$$

Set

$$A_{m,r} = \frac{s}{m^\nu \Gamma(\nu+1)}, \quad B_{m,r} = \frac{r}{m^{\nu+1} \Gamma(\nu+2)}. \quad (32)$$

Then, uniformly in $0 \leq r < m$,

$$d_{\text{TV}}\left(\frac{\text{Tr}(U_N^m)}{\sqrt{m}}, Z\right) = \frac{1}{2} \int_0^\infty e^{-u} |(-1)^q A_{m,r} L_\nu(u) + (-1)^{q+1} B_{m,r} L_{\nu+1}(u)| du + o_q\left(\frac{s+1}{m^\nu}\right). \quad (33)$$

In particular, if $r/m \rightarrow \theta \in [0, 1)$, then

$$d_{\text{TV}}\left(\frac{\text{Tr}(U_N^m)}{\sqrt{m}}, Z\right) \sim \frac{1-\theta}{2m^q \Gamma(q+2)} \int_0^\infty e^{-u} |L_{q+1}(u)| du. \quad (34)$$

If instead $s = m - r \rightarrow c \in \mathbb{N}$, then

$$d_{\text{TV}}\left(\frac{\text{Tr}(U_N^m)}{\sqrt{m}}, Z\right) \sim \frac{1}{2m^{q+1}} \int_0^\infty e^{-u} \left| \frac{c}{\Gamma(q+2)} L_{q+1}(u) - \frac{1}{\Gamma(q+3)} L_{q+2}(u) \right| du. \quad (35)$$

Theorem 6 (Local Poisson approximation). *Let U_N be Haar distributed on $U(N)$. For $N, m \geq 1$, write the quotient and remainder as in (??), and, for $j \geq 2$, set*

$$\gamma_{j;N,m} = \frac{r(q+1)^j + (m-r)q^j}{N^j}, \quad \varepsilon_{N,m} = \frac{q + \mathbf{1}_{\{r>0\}}}{N}. \quad (36)$$

Thus $\gamma_{2;N,m} = \beta_{N,m}$ from (21). Fix $L > 0$ and an interval $B \subset \mathbb{T}_N$ of length L . Let $\Xi_{N,m}$ and Π_B be as in (19)–(20), let $Z_L \sim \text{Pois}(L)$, and use the constant $A(L)$ defined in (22). There is a constant $C_L < \infty$ such that, whenever $N > L$ and $L\varepsilon_{N,m} \leq \frac{1}{4}$,

$$\begin{aligned} & \left| d_{\text{TV}}(\mathcal{L}(\Xi_{N,m} \upharpoonright_B), \mathcal{L}(\Pi_B)) - A(L)\beta_{N,m} \right| \\ & + \left| d_{\text{TV}}(\mathcal{L}(\Xi_{N,m}(B)), \mathcal{L}(Z_L)) - A(L)\beta_{N,m} \right| \leq C_L (\gamma_{3;N,m} + \beta_{N,m}^2 + \gamma_{4;N,m}). \end{aligned} \quad (37)$$

Here we regard $\Xi_{N,m} \upharpoonright_B$ and Π_B as random elements of the configuration space $\mathcal{N}(B)$ of finite simple counting measures on B . Consequently, if $N \rightarrow \infty$ and $m = m_N \rightarrow \infty$, with no restriction on their relative rates, each of the two total-variation distances in equation (37) is asymptotic to $A(L)\beta_{N,m}$.

Remark 7 (Normalisation). Equation (14) fixes the unit-variance normalisation. It contributes $m^{-1/2}$ in the first three trace theorems, where $m < N$, and $N^{-1/2}$ when $m \geq N$. The change marks the point at which every Rains block becomes one-dimensional; below it, each nontrivial block still contains CUE repulsion.

Remark 8 (Why the quotient remainder matters). The integer $r = N - m\lfloor N/m \rfloor$ records how many Rains blocks have size $q+1$ rather than q . Away from $r = m$, their first correction is one order smaller. Near that endpoint, however, $m-r$ may be bounded and the two adjacent Laguerre modes have comparable size; this is why theorem 5 retains both. The same block-size data enter the local theorem through $\gamma_{j;N,m}$.

Remark 9. Courteaut, Johansson and Lambert [5] prove sharp-order bounds which interpolate between the fixed-power and high-power regimes. In particular, their upper bound for fixed p is of order $q^{1/4} \sqrt{\log N} / (p^q \Gamma(q+2))$, whereas theorem 1 identifies the leading term, which is smaller by a factor of order $q^{-1/2} (\log N)^{-1/2}$. For $m_N \geq N$, their $O(N^{-1})$ estimate is sharpened by theorem 3 to an explicit constant and an $O(N^{-2})$ remainder.

2.1 Structure of the leading corrections

We prove each global theorem at the density level. If $f_{N,m}$ denotes the density of the normalised trace, the common target is an expansion

$$f_{N,m} = g + \Lambda_{N,m} + r_{N,m}, \quad \|r_{N,m}\|_1 = o(\|\Lambda_{N,m}\|_1). \quad (38)$$

The reverse triangle inequality then gives

$$d_{\text{TV}}(f_{N,m}, g) = \frac{1}{2} \|\Lambda_{N,m}\|_1 + o(\|\Lambda_{N,m}\|_1). \quad (39)$$

The signed function $\Lambda_{N,m}$ is not itself a probability density. It has integral zero and describes where the correction removes mass from, and adds it to, the Gaussian approximation. Because total variation discards the sign only after assembling this coherent correction, the order of operations is essential: bounding individual moment or determinant terms before their cancellation would obscure the leading scale.

In the fixed-power and joint-growth theorems, the leading correction is a single Laguerre profile after the first band has been summed:

$$\Lambda_{N,m}(x) \approx A_{N,m} g(x) L_\nu(|x|^2).$$

Radial integration changes its $L^1(\mathbb{R}^2)$ norm into $A_{N,m} \int_0^\infty e^{-u} |L_\nu(u)| du$. The integral form in the theorem is therefore the more informative finite-parameter expression: it retains the exact oscillatory profile. The second equivalence uses the universal asymptotic

$$\int_0^\infty e^{-u} |L_\nu(u)| du \sim \frac{2^{7/4} \Gamma(3/4)}{\pi^{3/2}} \nu^{-1/4}$$

and is the more convenient headline rate. We state both because neither is redundant: the first identifies the actual first correction, while the second makes the dependence on N, m explicit.

The fixed-quotient formula retains two adjacent Laguerre modes. If $r/m \rightarrow \theta < 1$, the coefficient of L_{q+2} is lower order and the one-mode simplification (34) follows. If $m-r$ stays bounded, both coefficients are of order $m^{-(q+1)}$ and may cancel on parts of the radial line. The absolute value must therefore remain outside their sum, as in (35). This is also why a bound on the two coefficients separately would not determine the sharp constant uniformly in the remainder r .

The four global statements exhaust the natural ways in which the quotient $q = \lfloor N/m \rfloor$ can behave: m fixed (hence $q \rightarrow \infty$), q fixed (hence $m \asymp N$), both $m, q \rightarrow \infty$, and $q = 0$ when $m \geq N$. Their formulas meet in order of magnitude, but a single uniform leading expression would have to resolve different collections of Laguerre modes near the boundaries. Keeping the regimes separate makes the source of each constant visible.

2.2 Exact mixed trace moments: single-frequency and multitime factorisations

Several useful representations describe the same average, and the later proofs deliberately move between them. The Haar integral is the probabilistic definition, and Heine–Szegő turns its exponential generating function into a Toeplitz determinant. Expanding in Schur functions gives a character truncation which, after Charlier diagonalisation, reveals the stable identity kernel and the successive exterior bands. Differentiating the Toeplitz determinant instead gives the finite displacement formula, an efficient route to exact mixed exponents. Finally, the Borodin–Okounkov formula converts the normalised Toeplitz determinant into a Fredholm determinant; that global object is best for estimating the sum of all higher bands.

Schematically, the roles are

$$\begin{array}{ll} \text{Charlier–Ewens kernel} & : \text{ stable term and exterior bands,} \\ \text{displacement determinants} & : \text{ exact coefficients,} \\ \text{Fredholm determinant} & : \text{ global remainder control.} \end{array} \quad (40)$$

No one representation dominates the others. In particular, the Charlier–Ewens formula identifies the first correction sharply but is signed, so taking absolute values inside its convolution is too expensive. The Fredholm determinant supplies the cancellation needed for analytic bounds, while the displacement formula proves the residue-class block structure without importing the spectral power-map theorem.

We extend the notation in (??) to vectors of possibly unequal lengths. For positive integer vectors $\mathbf{p} = (p_1, \dots, p_{k_1})$ and $\mathbf{q} = (q_1, \dots, q_{k_2})$, put

$$C_N(\mathbf{p}, \mathbf{q}) := \mathbb{E}_{U(N)} \left[\prod_{\alpha=1}^{k_1} \text{Tr}(U^{p_\alpha}) \prod_{\beta=1}^{k_2} \overline{\text{Tr}(U^{q_\beta})} \right]. \quad (41)$$

Multiplication of U by a scalar phase shows immediately that this moment vanishes unless $\sum_{\alpha} p_{\alpha} = \sum_{\beta} q_{\beta}$. We begin with the balanced case because here we can separate the stable Wick term and its correction canonically.

The Charlier–Ewens formula organises the global correction by the number of shifted diagonal points of a partition lying beyond the finite- N length cutoff. We call these level sets exterior bands after defining the counting function below. Assume from now on that

$$D := \sum_{\alpha=1}^{k_1} p_{\alpha} = \sum_{\beta=1}^{k_2} q_{\beta},$$

and let $\mu, \nu \vdash D$ be the partitions having parts $p_1, \dots, p_{k_1}$ and $q_1, \dots, q_{k_2}$, respectively. Write

$$\mu = (1^{a_1} 2^{a_2} \dots), \quad \nu = (1^{b_1} 2^{b_2} \dots), \quad z_{\mu} = \prod_{j \geq 1} j^{a_j} a_j!.$$

Let $P_n(x; \alpha)$ denote the monic Charlier polynomial

$$\sum_{n=0}^{\infty} P_n(x; \alpha) \frac{t^n}{n!} = e^{-\alpha t} (1+t)^x. \quad (42)$$

Our normalisation follows the standard Poisson-orthogonal convention in [18]. For a finitely supported multi-index $\mathbf{c} = (c_1, c_2, \dots)$, set

$$\text{wt}(\mathbf{c}) = \sum_{j \geq 1} j c_j, \quad \Phi_{\mathbf{c}}(\mu) = \prod_{j \geq 1} \frac{P_{c_j}(a_j; 1/j)}{\sqrt{j^{-c_j} c_j!}}.$$

Define

$$K_{N,D}^{\text{Ch}}(\mu, \nu) = \sum_{\text{wt}(\mathbf{c}) < D-N} \Phi_{\mathbf{c}}(\mu) \Phi_{\mathbf{c}}(\nu). \quad (43)$$

For kernels on partitions of D , their Ewens convolution is

$$(A \star_D B)(\mu, \nu) = \sum_{\gamma \vdash D} \frac{A(\mu, \gamma) B(\gamma, \nu)}{z_{\gamma}}, \quad (44)$$

whose identity is $I_D(\mu, \nu) = z_{\mu} \mathbf{1}_{\{\mu=\nu\}}$. Finally, define

$$\begin{aligned} \binom{K}{0}_{\star_D} &= I_D, \\ \binom{K}{\ell}_{\star_D} &= \frac{1}{\ell!} K \star_D (K - I_D) \star_D \dots \star_D (K - (\ell-1)I_D), \quad \ell \geq 1. \end{aligned} \quad (45)$$

The reason for the Charlier basis is structural. Under the Ewens measure z_{γ}^{-1} , the multiplicities of cycles of length j are the Poisson coordinates with parameter $1/j$; the centred falling factorials of these coordinates are precisely the Charlier polynomials. The next finite identity identifies their truncated reproducing kernel with the number of shifted diagonal points of a partition.

Theorem 10 (Length-exceedance diagonalisation). *For $\lambda \vdash D$ set*

$$d_N(\lambda) = \#\{i \geq 1 : \lambda_i \geq N+i\}, \quad (46)$$

equivalently, $d_N(\lambda) = \max(\{0\} \cup \{\ell \geq 1 : \lambda_{\ell} \geq N+\ell\})$. These definitions agree because $i \mapsto \lambda_i - i$ is strictly decreasing. Then

$$K_{N,D}^{\text{Ch}}(\mu, \nu) = \sum_{\lambda \vdash D} d_N(\lambda) \chi_{\mu}^{\lambda} \chi_{\nu}^{\lambda}. \quad (47)$$

Proof. We give a coefficient proof using Bernstein operators. This avoids choosing or ordering rim-hook removals: the inclusion–exclusion in each centred Charlier factor is performed first, by a translation operator.

Let Λ be the ring of symmetric functions with its Hall inner product. Thus

$$e_\gamma := \frac{p_\gamma}{\sqrt{z_\gamma}}, \quad \gamma \vdash w, \quad w \geq 0,$$

is an orthonormal power-sum basis, and the Schur functions are another orthonormal basis. Frobenius's formula [19, Chapter I] reads

$$s_\lambda = \sum_{\gamma \vdash D} \frac{\chi_\gamma^\lambda}{\sqrt{z_\gamma}} e_\gamma. \quad (48)$$

Denote by Π_w the orthogonal projection onto the homogeneous component Λ_w .

The local Charlier transform follows directly from [equation \(42\)](#): multiply by $x^a/a!$, sum over a , and compare the coefficient of t^c . This gives

$$\sum_{a=0}^{\infty} P_c(a; 1/j) \frac{x^a}{a!} = e^x (x - 1/j)^c. \quad (49)$$

For a multi-index $\mathbf{c}$, let $\rho(\mathbf{c}) = (1^{c_1} 2^{c_2} \dots)$ and $w = \text{wt}(\mathbf{c})$. Substituting $x = p_j/j$ in [equation \(49\)](#), multiplying over j , and then taking homogeneous degree D yields the exact Frobenius series

$$\begin{aligned} F_{\mathbf{c}}^{(D)} &:= \sum_{\gamma \vdash D} \Phi_{\mathbf{c}}(\gamma) \frac{p_\gamma}{z_\gamma} \\ &= \Pi_D \left[\exp \left(\sum_{j \geq 1} \frac{p_j}{j} \right) \prod_{j \geq 1} \frac{(p_j - 1)^{c_j}}{\sqrt{j^{c_j} c_j!}} \right]. \end{aligned} \quad (50)$$

All Charlier inclusion–exclusion terms are contained in the translations $p_j \mapsto p_j - 1$ in this identity.

Write $p_j^\perp = j \partial / \partial p_j$ and introduce the Bernstein vertex operator [19, Chapter I, Section 5]

$$B(z) = \exp \left(\sum_{j \geq 1} \frac{p_j z^j}{j} \right) \exp \left(- \sum_{j \geq 1} \frac{p_j^\perp z^{-j}}{j} \right) = \sum_{n \in \mathbb{Z}} B_n z^n. \quad (51)$$

At $z = 1$ the second exponential is $\exp(-\sum_j \partial / \partial p_j)$, hence it performs precisely the translation $p_j \mapsto p_j - 1$. Since $e_{\rho(\mathbf{c})} = \prod_j p_j^{c_j} / \sqrt{j^{c_j} c_j!}$, [equation \(50\)](#) becomes

$$F_{\mathbf{c}}^{(D)} = \Pi_D B(1) e_{\rho(\mathbf{c})} = B_{D-w} e_{\rho(\mathbf{c})}. \quad (52)$$

The last equality is simply homogeneous-degree bookkeeping; in particular, the formal specialisation $z = 1$ causes no convergence issue.

We now sum the rank-one operators associated with the left side of [equation \(47\)](#). Put $T = D - N$. Completeness of the normalised power sums in every Λ_w and [equation \(52\)](#) give

$$\begin{aligned} \mathcal{A}_{N,D} &:= \sum_{\substack{\mathbf{c} \\ \text{wt}(\mathbf{c}) < T}} |F_{\mathbf{c}}^{(D)} \rangle \langle F_{\mathbf{c}}^{(D)}| \\ &= \sum_{w=0}^{T-1} B_{D-w} \Pi_w B_{D-w}^*. \end{aligned} \quad (53)$$

It remains to diagonalise this finite operator. We include that step. The defining identity for Bernstein modes is

$$B_n s_\eta = s_{(n, \eta)},$$

where the right side is a generalised Schur function. The elementary straightening relation

$$s_{(\dots, a, b, \dots)} = -s_{(\dots, b-1, a+1, \dots)}$$

shows the following. If $\lambda \vdash D$ and $i \geq 1$ is a row of λ , set

$$n_i = \lambda_i - i + 1, \quad \eta^{(i)} = (\lambda_1 + 1, \dots, \lambda_{i-1} + 1, \lambda_{i+1}, \lambda_{i+2}, \dots). \quad (54)$$

Then $\eta^{(i)} \vdash D - n_i$ and

$$B_{n_i} s_{\eta^{(i)}} = (-1)^{i-1} s_\lambda. \quad (55)$$

Conversely, straightening (n, η) to λ is possible only for the unique row i with $n = n_i$; the integers n_i are strictly decreasing. Taking adjoints in [equation \(55\)](#), and using orthonormality of Schur functions, therefore gives

$$B_n \Pi_{D-n} B_n^* s_\lambda = \mathbf{1}_{\{n = \lambda_i - i + 1 \text{ for some } i\}} s_\lambda, \quad n > 0. \quad (56)$$

In [equation \(53\)](#), $n = D - w$ runs through $N + 1, \dots, D$. Hence

$$\begin{aligned} \mathcal{A}_{N,D} s_\lambda &= \#\{i : \lambda_i - i + 1 > N\} s_\lambda \\ &= \#\{i : \lambda_i \geq N + i\} s_\lambda = d_N(\lambda) s_\lambda. \end{aligned} \quad (57)$$

Finally, the (e_μ, e_ν) matrix entry of [equation \(53\)](#) is $K_{N,D}^{\text{Ch}}(\mu, \nu) / \sqrt{z_\mu z_\nu}$, whereas [equations \(48\)](#) and [\(57\)](#) show that it is

$$\frac{1}{\sqrt{z_\mu z_\nu}} \sum_{\lambda \vdash D} d_N(\lambda) \chi_\mu^\lambda \chi_\nu^\lambda.$$

Multiplication by $\sqrt{z_\mu z_\nu}$ proves [equation \(47\)](#). $\square$

Theorem 11 (Charlier–Ewens correction). *Let*

$$s(N, D) = \max\{\ell \geq 1 : \ell(N + \ell) \leq D\}, \quad (58)$$

with $s(N, D) = 0$ if the set is empty. Then

$$C_N(\mathbf{p}, \mathbf{q}) = z_\mu \mathbf{1}_{\{\mu=\nu\}} + \mathcal{R}_N(\mu, \nu), \quad (59)$$

where

$$\mathcal{R}_N(\mu, \nu) = (-1)^{k_1+k_2+1} \sum_{\ell=1}^{s(N,D)} (-1)^{\ell+1} \left(K_{N,D}^{\text{Ch}} \right)_\ell^{\star_D}(\mu, \nu). \quad (60)$$

Thus the Diaconis–Shahshahani term is the identity kernel and the ℓ th summand is the ℓ th exterior band.

Proof. The Frobenius characteristic formula and Schur orthogonality give the standard character truncation used by Diaconis and Shahshahani [\[8\]](#):

$$C_N(\mu, \nu) = \sum_{\substack{\lambda \vdash D \\ \ell(\lambda) \leq N}} \chi_\mu^\lambda \chi_\nu^\lambda. \quad (61)$$

The full character orthogonality relation is $\sum_{\lambda \vdash D} \chi_\mu^\lambda \chi_\nu^\lambda = z_\mu \mathbf{1}_{\{\mu=\nu\}}$; hence $\mathcal{R}_N$ is minus the contribution from $\ell(\lambda) > N$.

Conjugate every excluded partition. By $\chi_\mu^{\lambda'} = (-1)^{D-k_1} \chi_\mu^\lambda$, this contributes the common sign $(-1)^{k_1+k_2}$. The excluded condition becomes $d_N(\lambda) \geq 1$.

We now carry out the convolutional Cauchy–Binet step. Character orthogonality and [theorem 10](#) imply

$$(K_{N,D}^{\text{Ch}})^{\star_D J}(\mu, \nu) = \sum_{\lambda \vdash D} d_N(\lambda)^j \chi_\mu^\lambda \chi_\nu^\lambda, \quad (62)$$

$$\left(K_{N,D}^{\text{Ch}} \right)_\ell^{\star_D}(\mu, \nu) = \sum_{\lambda \vdash D} \binom{d_N(\lambda)}{\ell} \chi_\mu^\lambda \chi_\nu^\lambda. \quad (63)$$

Thus Ewens convolution is not a formal analogy: it is ordinary matrix multiplication after diagonalisation by the character table, and the falling factorial in [equation \(45\)](#) becomes the ordinary binomial coefficient in [equation \(63\)](#). Finally,

$$\sum_{\ell \geq 1} (-1)^{\ell+1} \binom{d}{\ell} = \mathbf{1}_{\{d \geq 1\}}.$$

Moreover, $d_N(\lambda) \geq \ell$ forces $\lambda_\ell \geq N + \ell$ and hence $D = |\lambda| \geq \ell(N + \ell)$; therefore the sum stops at $s(N, D)$. Substituting [equation \(63\)](#) and the last binomial identity into the exterior part of [equation \(61\)](#), including its initial minus sign, proves [equation \(60\)](#). $\square$

Stable range and the first exterior band. The finite upper limit in [\(60\)](#) immediately gives two useful simplifications. If $D \leq N$, then $s(N, D) = 0$ and

$$C_N(\mu, \nu) = z_\mu \mathbf{1}_{\{\mu=\nu\}}. \quad (64)$$

For equal numbers of holomorphic and antiholomorphic traces, the right side is exactly the Wick sum in [\(12\)](#). Indeed, if $\mu = (1^{a_1} 2^{a_2} \dots)$, a Wick matching must pair equal exponents; it has $\prod_j a_j!$ choices and covariance product $\prod_j j^{a_j}$, giving z_μ . Thus [\(64\)](#) recovers the unitary Diaconis–Shahshahani moment identity directly from the absence of any length exceedance.

In the next range,

$$N + 1 \leq D < 2(N + 2), \quad (65)$$

one has $s(N, D) = 1$, so the full correction is linear:

$$\mathcal{R}_N(\mu, \nu) = (-1)^{k_1+k_2+1} K_{N,D}^{\text{Ch}}(\mu, \nu). \quad (66)$$

This is the widest degree interval on which no second exterior band can occur. It should not be confused with a statement that [\(66\)](#) is numerically linear in $D - N$; rather, it is linear in the truncated kernel, with no convolutional minor of order at least two. The distinction matters when several trace exponents are present.

For the single-power moment $\mathbb{E}|\text{Tr}(U^p)|^{2k}$, one has $D = kp$. Hence all moments through $k \leq \lfloor N/p \rfloor$ have their Gaussian values $k!p^k$, and the first possible defect is

$$k = \left\lfloor \frac{N}{p} \right\rfloor + 1. \quad (67)$$

The statement is familiar from the Diaconis–Shahshahani range and also follows from Rains’s block decomposition. What the Charlier kernel adds is an explicit matrix element for the defect and, through its convolutional binomial, an exact organisation of every subsequent exterior band. This is the information used in the near-edge cumulant calculation below.

Relation to Soshnikov's cumulants. Soshnikov's formula converts the exponential trace generating function into cumulants and expresses each one as a signed sum of interval-overlap lengths, whose cancellations give a short route to Gaussian limits [24]. Equation (60) instead separates the complete Wick moment and grades the remaining moment kernel by $d_N(\lambda)$, a form better suited to the coefficient and remainder estimates below.

A direct computational formula. For direct computation, the following finite determinant is often faster than the Charlier–Ewens formula: it replaces the character sum by signed partial permutation determinants and requires no balance assumption. It follows directly from the Heine–Szegő determinant by multilinearity; to our knowledge, no previous source writes it in this explicit form.

Theorem 12 (Displacement determinant). *For $\mathbf{r} \in \{0, \dots, N-1\}^{k_1}$ and $\mathbf{s} \in \{0, \dots, N-1\}^{k_2}$, define*

$$h_a(\mathbf{r}, \mathbf{s}) = \sum_{\alpha=1}^{k_1} p_\alpha \mathbf{1}_{\{r_\alpha=a\}} - \sum_{\beta=1}^{k_2} q_\beta \mathbf{1}_{\{s_\beta=a\}}, \quad 0 \leq a \leq N-1. \quad (68)$$

Then

$$C_N(\mathbf{p}, \mathbf{q}) = \sum_{\mathbf{r}, \mathbf{s}} \det[\mathbf{1}_{\{b=a-h_a(\mathbf{r}, \mathbf{s})\}}]_{a,b=0}^{N-1}. \quad (69)$$

In particular, every summand is 0, 1, or -1 .

Proof. Introduce independent variables t_α, u_β and the symbol

$$\omega(z) = \exp\left(\sum_{\alpha=1}^{k_1} t_\alpha z^{p_\alpha} + \sum_{\beta=1}^{k_2} u_\beta z^{-q_\beta}\right).$$

Set $\omega_k = [z^{-k}] \omega(z)$. The Heine–Szegő identity [25, 4] writes the corresponding exponential trace average as

$$\mathbb{E}_{U(N)} \exp\left(\sum_{\alpha} t_\alpha \operatorname{Tr}(U^{p_\alpha}) + \sum_{\beta} u_\beta \overline{\operatorname{Tr}(U^{q_\beta})}\right) = \det[\omega_{b-a}]_{a,b=0}^{N-1}.$$

Differentiate once in every t_α and u_β at the origin. Multilinearity assigns each derivative to a row. If row a receives the positive derivatives indexed by $r_\alpha = a$ and the negative derivatives indexed by $s_\beta = a$, it becomes $[\mathbf{1}_{\{b=a-h_a(\mathbf{r}, \mathbf{s})\}}]_b$. Summing over all assignments gives equation (69). $\square$

2.2.1 Bessel and Fredholm specialisations

Let

$$M_{N,p}(k) = \mathbb{E}_{U(N)} |\operatorname{Tr}(U^p)|^{2k}$$

and define the entire Bessel determinant

$$H_d(z) = \det[I_{a-b}(2\sqrt{z})]_{a,b=0}^{d-1}, \quad H_0(z) = 1. \quad (70)$$

Although individual entries use $\sqrt{z}$, the determinant is entire. This Bessel determinant is the one-time Toeplitz specialisation appearing in Gessel's symmetric-function identity [14].

Corollary 13 (Trace-level block factorisation). *With $N = pq + r$ as in equation (24),*

$$\sum_{k=0}^{\infty} M_{N,p}(k) \frac{z^k}{(k!)^2} = H_{q+1}(z)^r H_q(z)^{p-r}. \quad (71)$$

Consequently, for $X_{N,p} = p^{-1/2} \text{Tr}(U_N^p)$ and $a = |t|^2/4$,

$$\mathbb{E} e^{it \cdot X_{N,p}} = H_{q+1}(-a/p)^r H_q(-a/p)^{p-r}. \quad (72)$$

Proof. In equation (69), take k positive and k negative copies of the same exponent p . If A_i and B_i are the numbers of positive and negative labels assigned to row i , division by $(k!)^2$ and summation in k give

$$\sum_{\mathbf{A}, \mathbf{B} \geq 0} \prod_{i=0}^{N-1} \frac{z^{(A_i+B_i)/2}}{A_i! B_i!} \det[\mathbf{1}_{\{j=i-p(A_i-B_i)\}}]_{i,j}.$$

The determinant already forces $\sum_i A_i = \sum_i B_i$. By multilinearity, the (i, j) entry after summation is zero unless $i \equiv j \pmod{p}$, and otherwise is

$$\sum_{A-B=(i-j)/p} \frac{z^{(A+B)/2}}{A! B!} = I_{(i-j)/p}(2\sqrt{z}).$$

Reorder the indices by residue class modulo p . There are r classes of size $q+1$ and $p-r$ classes of size q , which proves equation (71). Substitution of $z = -a/p$ gives equation (72). This argument does not use an eigenvalue power-map theorem. Rains's theorem [22, 23] gives the same factorisation at the level of the full spectrum; here it follows directly from the displacement determinant. $\square$

The first altered single-power moment. Corollary 13 makes the first defect completely explicit. The character truncation agrees with e^z through degree d ; at degree $d+1$ it excludes only the partition (1^{d+1}) , whose character at the identity is one. Consequently the first logarithmic correction is

$$\log H_d(z) = z - \frac{z^{d+1}}{((d+1)!)^2} + O(z^{d+2}), \quad (73)$$

the product (71), with $v = q+1$, gives

$$\mathbb{E} |\text{Tr}(U_N^p)|^{2k} = k! p^k \quad (0 \leq k \leq q), \quad \mathbb{E} |\text{Tr}(U_N^p)|^{2v} = v! p^v - (p-r). \quad (74)$$

Indeed, the $p-r$ blocks of size q each contribute the coefficient $-1/(v!)^2$ in (73); the r larger blocks have no defect at that degree. Multiplication by $(v!)^2$ converts the generating coefficient back into the moment. Thus the first deviation is an integer count of the smaller residue classes. For $p=1$ this reduces to the familiar identity $\mathbb{E} |\text{Tr}(U_N)|^{2(N+1)} = (N+1)! - 1$.

This example separates three levels of information. The stable-range theorem locates the first possible defect. Rains's decomposition explains why the number $p-r$ of smaller blocks should occur. The determinant factorisation computes its exact coefficient in one line and continues to every higher moment by coefficient extraction. The total-variation problem is subtler: after normalisation, all fixed shifts in the first band contribute at the same leading order, so (74) alone does not produce the factor $e^{2/p}$.

The scalar Bessel product is the one-time specialisation of a multitime factorisation. We record the extension because it links the displacement determinant directly to the standard truncated Cauchy object.

Corollary 14 (Multitime factorisation and Fredholm correction). *Let $g \geq 1$, write $N = gQ + R$ with $0 \leq R < g$, and suppose that $\mathbf{t} = (t_k)_{k \geq 1}$ and $\mathbf{s} = (s_k)_{k \geq 1}$ satisfy*

$$\sum_{k \geq 1} k(|t_k| + |s_k|) < \infty. \quad (75)$$

Define

$$\mathcal{H}_d(\mathbf{t}, \mathbf{s}) = \mathbb{E}_{V \in \mathcal{U}(d)} \exp \left\{ \sum_{k \geq 1} t_k \operatorname{Tr}(V^k) + \sum_{k \geq 1} s_k \operatorname{Tr}(V^{-k}) \right\}, \quad \mathcal{H}_0 = 1. \quad (76)$$

Then

$$\begin{aligned} \mathcal{Z}_{N,g}(\mathbf{t}, \mathbf{s}) &:= \mathbb{E}_{U \in \mathcal{U}(N)} \exp \left\{ \sum_{k \geq 1} t_k \operatorname{Tr}(U^{gk}) + \sum_{k \geq 1} s_k \operatorname{Tr}(U^{-gk}) \right\} \\ &= \mathcal{H}_{Q+1}(\mathbf{t}, \mathbf{s})^R \mathcal{H}_Q(\mathbf{t}, \mathbf{s})^{g-R}. \end{aligned} \quad (77)$$

Put

$$\Phi(z) = \exp \left(\sum_{k \geq 1} t_k z^k + \sum_{k \geq 1} s_k z^{-k} \right), \quad \mathcal{I}_n = [z^{-n}] \Phi(z), \quad (78)$$

$$\mathfrak{s}(\mathbf{t}, \mathbf{s}) = \sum_{k \geq 1} k t_k s_k, \quad \mathcal{F}_d(\mathbf{t}, \mathbf{s}) = e^{-\mathfrak{s}(\mathbf{t}, \mathbf{s})} \mathcal{H}_d(\mathbf{t}, \mathbf{s}). \quad (79)$$

The one-block factor and its normalised correction are

$$\mathcal{H}_d = \det[\mathcal{I}_{b-a}]_{a,b=0}^{d-1}, \quad \mathcal{F}_d = \det(I - K_d). \quad (80)$$

Here, with Laurent coefficients denoted by $[z^n]$, set

$$b(z) = \exp \left(\sum_{k \geq 1} s_k z^{-k} - t_k z^k \right), \quad c(z) = b(z)^{-1}, \quad (81)$$

let $\operatorname{Hank}(a) = ([z^{i+j+1}]a(z))_{i,j \geq 0}$ be the Hankel operator, write $\mathfrak{d}(z) = c(z^{-1})$, and let Q_d project $\ell^2(\mathbb{Z}_{\geq 0})$ onto $\ell^2(\{d, d+1, \dots\})$. Then

$$K_d = Q_d \operatorname{Hank}(b) \operatorname{Hank}(\mathfrak{d}) Q_d. \quad (82)$$

In particular, on the direct sum of g tail spaces,

$$e^{-g\mathfrak{s}} \mathcal{Z}_{N,g} = \det(I - \mathbf{K}_{N,g}), \quad (83)$$

$$\mathbf{K}_{N,g} = K_{Q+1}^{\oplus R} \oplus K_Q^{\oplus (g-R)}, \quad (84)$$

$$\det(I - \mathbf{K}_{N,g}) = 1 - \operatorname{Tr} \mathbf{K}_{N,g} + \sum_{\ell \geq 2} (-1)^\ell \operatorname{Tr}(\wedge^\ell \mathbf{K}_{N,g}), \quad (85)$$

$$\operatorname{Tr} \mathbf{K}_{N,g} = R \operatorname{Tr} K_{Q+1} + (g - R) \operatorname{Tr} K_Q. \quad (86)$$

All identities also hold coefficientwise in the completed formal power-series ring with $\deg t_k = \deg s_k = k$.

Proof. The symbol for $\mathcal{Z}_{N,g}$ is $\Phi(z^g)$. Its Fourier coefficients vanish outside $g\mathbb{Z}$, so reordering the Toeplitz indices by residue class modulo g produces R blocks of size $Q+1$ and $g-R$ blocks of size Q . This proves (77). Heine–Szegő gives the Toeplitz determinant in (80).

The Borodin–Okounkov–Geronimo–Case identity, in the product-of-Hankel normalisation of [4, 3], gives

$$\det[\mathcal{I}_{b-a}]_{a,b=0}^{d-1} = \exp\left(\sum_{k \geq 1} k t_k s_k\right) \det(I - Q_d \text{Hank}(b) \text{Hank}(d) Q_d).$$

Condition (75) makes the Hankel product trace class and justifies all rearrangements. Taking the product over residue classes gives (83) and (84). The Fredholm exterior-power expansion gives (85), and additivity of the trace on direct sums gives (86). For the formal statement, truncate the times; every coefficient of fixed weighted degree uses only finitely many variables, after which the analytic identity applies coefficientwise. This also shows why no finite-support assumption on the original time sequences is needed. $\square$

The content of Corollary 14 is stronger than a Schur expansion of the truncated Cauchy sum. The scalar e^{g^5} is the infinite-dimensional, or stable, contribution; one operator $\mathbf{K}_{N,g}$ carries all finite- N information. Equation (85) separates its first trace from the exterior powers of order at least two, while (86) preserves the two block sizes forced by the remainder R . The fixed-power proof uses the first trace to identify the Laguerre band and the full determinant to control what remains. The fixed-quotient and joint-growth proofs use the same separation with a growing number of blocks. Thus the Fredholm form is not an unused reformulation: it is the analytic counterpart of the Charlier exterior-band expansion.

For finite Miwa alphabets, $t_k = p_k(X)/k$ and $s_k = p_k(Y)/k$ satisfy (75) whenever the alphabet entries lie strictly inside the unit disc. Hence the corollary includes the normalised truncated Cauchy object as a specialisation, without defining it by an immediately tautological Schur sum. For general time sequences, the weighted summability condition is the precise analytic hypothesis; finite support is unnecessary.

3 Analytic tools for total-variation asymptotics

Two complementary tools connect the exact generating functions to total variation. Fourier–Laguerre transfer identifies the leading signed density correction, while the Bessel Fredholm estimate controls the determinant remainder uniformly.

3.1 Fourier–Laguerre transfer

For $f \in L^1(\mathbb{R}^2)$, use

$$(\mathcal{F}f)(t) = \int_{\mathbb{R}^2} e^{it \cdot x} f(x) dx, \quad (\mathcal{F}^{-1}h)(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-it \cdot x} h(t) dt. \quad (87)$$

Then $(\mathcal{F}g)(t) = e^{-|t|^2/4}$. The ordinary Laguerre polynomials are

$$L_k(u) = \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{u^j}{j!}.$$

Lemma 15 (Laguerre modes). *For every $k \geq 0$,*

$$\mathcal{F}(L_k(|\cdot|^2)g)(t) = e^{-|t|^2/4} \frac{(|t|^2/4)^k}{k!}. \quad (88)$$

Moreover, if $R \in L^2(\mathbb{R}^2)$ and its distributional Laplacian ΔR belongs to $L^2(\mathbb{R}^2)$, and if $f = \mathcal{F}^{-1}R$, then

$$\|f\|_1 \leq C(\|R\|_2 + \|\Delta R\|_2). \quad (89)$$

Proof. The identity $\Delta^k g = (-4)^k k! L_k(|x|^2)g$ follows by induction from the Laguerre differential equation. Since $F(\Delta^k g) = (-1)^k |t|^{2k} e^{-|t|^2/4}$, with the signs cancelled by the chosen Laguerre convention, one obtains [equation \(88\)](#). For the second assertion,

$$\|f\|_1 \leq \|(1 + |x|^2)^{-1}\|_2 \|(1 + |x|^2)f\|_2 \leq C\|R - \Delta R\|_2,$$

by Cauchy–Schwarz and Plancherel. $\square$

We shall repeatedly use the following radial identity. If $a = |t|^2/4$ and $F = F(a)$ has two locally absolutely continuous derivatives, then, distributionally (and pointwise wherever the derivatives are classical),

$$\Delta_t F(a) = F'(a) + aF''(a). \quad (90)$$

For a rotationally invariant random variable X with the indicated moment finite, define its k th Laguerre coefficient

$$\beta_X(k) = \mathbb{E} L_k(|X|^2) = \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{\mathbb{E}|X|^{2j}}{j!}. \quad (91)$$

Whenever the resulting Bessel/moment series is absolutely convergent, expanding J_0 and using binomial inversion gives the characteristic-function identity

$$\mathbb{E} e^{it \cdot X} = e^{-a} \sum_{k=0}^{\infty} \beta_X(k) \frac{a^k}{k!}, \quad a = \frac{|t|^2}{4}. \quad (92)$$

All trace variables used below are bounded at fixed matrix dimension, so termwise expectation here follows directly from dominated convergence. If the characteristic function is integrable, Fourier inversion first gives the existence of a density. Whenever, in addition, the corresponding Laguerre series converges in L^1 , that density is

$$f_X(x) = g(x) \sum_{k=0}^{\infty} \beta_X(k) L_k(|x|^2). \quad (93)$$

3.2 Bessel Fredholm estimates

The fixed-power and joint-growth arguments use the same special linearisation of the Borodin–Okounkov determinant. We state it once in the normalisation of [equation \(94\)](#). This is important: the signed trace itself controls the quadratic error below, which does not follow from a generic trace-class estimate. On $\ell^2(\mathbb{N})$ let

$$\mathcal{J}(\rho) = [I_{i+j-1}(\rho)]_{i,j \geq 1}, \quad \mathcal{L} = \text{diag}((-1)^i)_{i \geq 1}, \quad \mathcal{K}(\rho) = -\mathcal{J}(\rho) \mathcal{L} \mathcal{J}(\rho) \mathcal{L}, \quad (94)$$

and let Q_d project onto the coordinates $d+1, d+2, \dots$

For $d \geq 1$ and $\rho \geq 0$, put

$$T_d(\rho) = \text{Tr}(Q_d \mathcal{K}(\rho) Q_d), \quad D_d(\rho) = \det(I - Q_d \mathcal{K}(\rho) Q_d). \quad (95)$$

Lemma 16 (Bessel determinant linearisation). *Let c_0 be the positive solution of $c_0 e^{1+c_0^2} = 1$, and put*

$$C_{\text{tr}} = \frac{1}{2\pi(1 - c_0^2)^2} = 0.2004129914 \dots < 0.201. \quad (96)$$

There is an absolute constant $C_1 < \infty$ such that, for every $d \geq 1$ and $0 \leq \rho \leq 2c_0(d+1)$,

$$|T_d(\rho)| \leq \frac{2 \exp(\rho^2/[2(d+1)])}{(1 - (\rho/[2(d+1)])^4)^2} \frac{(\rho/2)^{2d+2}}{\Gamma(d+2)^2}, \quad (97)$$

$$|T_d(\rho)| \leq \frac{C_{\text{tr}}}{d+1}, \quad (98)$$

$$\sum_{j \geq 2} \frac{1}{j} \left| \text{Tr}((Q_d \mathcal{K}(\rho) Q_d)^j) \right| \leq C_1 |T_d(\rho)|^2. \quad (99)$$

Consequently, with

$$\mathfrak{r}_d(\rho) = \sum_{j \geq 2} \frac{1}{j} \text{Tr}((Q_d \mathcal{K}(\rho) Q_d)^j), \quad (100)$$

one has

$$D_d(\rho) = \exp\{-T_d(\rho) - \mathfrak{r}_d(\rho)\}, \quad |\mathfrak{r}_d(\rho)| \leq C_1 |T_d(\rho)|^2, \quad (101)$$

$$D_d(\rho) = 1 - T_d(\rho) + E_d(\rho), \quad |E_d(\rho)| \leq C_1 |T_d(\rho)|^2. \quad (102)$$

The constants are uniform for the two adjacent dimensions $d = q, q+1$.

Proof. In the notation of Courteaut–Johansson–Lambert [5], their n is our d , their $r = |\zeta|$ is our ρ , and their Q_n has kernel $\text{span}(e_1, \dots, e_n)$, so it is exactly our tail projection Q_d . Their operator has the opposite overall sign from equation (94); this changes odd traces by a sign but does not affect any estimate in absolute value.

With this translation, equation (97) is their Lemma 2.6. Their Lemma 2.5 gives the sharper trace-norm estimate

$$\|Q_d \mathcal{K}(\rho) Q_d\|_1 \leq \frac{C_{\text{tr}}}{d+1} \leq \frac{0.201}{2} < 0.101 < 1 \quad (103)$$

throughout the stated disc, and hence also equation (98). Their Lemma 2.8 proves in the same range the special signed-trace estimate equation (99). The inequality $C_{\text{tr}} < 0.17$ printed alongside the exact definition in [5, Lemma 2.5] is numerically inconsistent with that definition; we use only the exact constant evaluated above.

For completeness, the remaining determinant step is short. Since the trace norm is smaller than one, Plemelj's expansion converges absolutely and gives

$$D_d = \exp \left\{ -T_d - \sum_{j \geq 2} \frac{1}{j} \text{Tr}((Q_d \mathcal{K} Q_d)^j) \right\}.$$

This is equation (101). If $z = T_d + \mathfrak{r}_d$, then $|z| \leq |T_d| + C_1 |T_d|^2$ is uniformly bounded by an absolute constant. The elementary inequality $|e^{-z} - 1 + z| \leq \frac{1}{2} |z|^2 e^{|z|}$, together with equation (99), yields

$$|D_d - 1 + T_d| \leq |e^{-z} - 1 + z| + |\mathfrak{r}_d| \leq C |T_d|^2.$$

Enlarging C_1 proves equation (102). This is also the content of Corollary 2.9 of [5] in the present notation. $\square$

4 Fixed powers: sharp Gaussian approximation

We now prove [theorem 1](#), retaining the notation (24); in particular, $q \rightarrow \infty$. The proof has four stages, each assigned to the representation for which it is simplest. The Charlier–Ewens formula identifies the complete first exterior band and shows that no lower degree can contribute. The one-block Bessel determinant computes the coefficients of that band and exposes their near-edge cumulants. The normalised Fredholm determinant controls the full characteristic function after the first trace is removed, including the unsmoothed case $p = 1$. Finally, Fourier–Laguerre transfer and spatial truncation turn the frequency estimate into the density expansion (38).

Separating these tasks is important. The Charlier convolution is exact but is not suited to taking absolute values term by term, whereas the determinant bound alone would not identify the coherent factor $e^{2/p}$. The phrase “first band” means exterior degree $\ell = 1$ in (60), not a single Taylor coefficient. For fixed p , that band contains degrees $\nu, \nu + 1, \dots, 2\nu - 1$. Although degree ν is the first non-Gaussian one, every fixed shift has the same leading Laguerre oscillation after normalisation. The coefficient ratios form an exponential series, which anticipates (154) and explains why the first defective moment alone does not determine the leading constant.

4.1 Near-edge cumulants and the first correction band

Define balanced cumulants $\kappa_k^{(d)}$ by

$$\log H_d(z) = z + \sum_{k \geq 2} \kappa_k^{(d)} \frac{z^k}{(k!)^2}. \quad (104)$$

Lemma 17 (Near-edge cumulants). *For $1 \leq s \leq d + 1$,*

$$\kappa_{d+s}^{(d)} = \frac{(-1)^s}{(s-1)!} (2d+s)_{s-1}, \quad (105)$$

where $(x)_j = x(x+1) \cdots (x+j-1)$. All $\kappa_k^{(d)}$ with $2 \leq k \leq d$ vanish.

Proof. Apply [theorem 11](#) to $\mu = \nu = (1^k)$. For $k \leq d$ the exterior sum is empty, giving the Gaussian moments. If $k = d + s \leq 2d + 1$, the logarithm of the moment series cannot contain a product of two non-Gaussian terms, because every such product has degree at least $2d + 2$. Hence only the first exterior trace contributes. Combining [equations \(123\) and \(124\)](#) below, or equivalently performing the one-coordinate Charlier extraction in [equation \(43\)](#), gives

$$\kappa_{d+s}^{(d)} = \sum_{h=1}^s (-1)^{s+h+1} h \frac{(2d+s+h+1)_{s-h}}{(s-h)!}. \quad (106)$$

We spell out the cancellation. Put $u = s - h$ and $n = 2d + 2s$; then

$$\frac{(2d+2s-u+1)_u}{u!} = \binom{n}{u}$$

and therefore

$$\kappa_{d+s}^{(d)} = \sum_{u=0}^{s-1} (-1)^{u+1} (s-u) \binom{n}{u}.$$

The two elementary partial-binomial identities

$$\begin{aligned}\sum_{u=0}^{s-1} (-1)^u \binom{n}{u} &= (-1)^{s-1} \binom{n-1}{s-1}, \\ \sum_{u=0}^{s-1} (-1)^u u \binom{n}{u} &= (-1)^{s-1} n \binom{n-2}{s-2}\end{aligned}$$

(the second follows from $u \binom{n}{u} = n \binom{n-1}{u-1}$) yield

$$\kappa_{d+s}^{(d)} = (-1)^s \binom{2d+2s-2}{s-1} = \frac{(-1)^s}{(s-1)!} (2d+s)_{s-1}.$$

□

For $X_{N,p} = p^{-1/2} \text{Tr}(U_N^p)$ write

$$\beta_{N,p}(k) = \mathbb{E} L_k(|X_{N,p}|^2), \quad A_{N,p} = \frac{p-r}{p^\nu \Gamma(\nu+1)}.$$

For $0 \leq j \leq q$, put

$$c_{q,j} = \frac{(q+1)!}{(q+j+1)!} \frac{(2q+j+1)_j}{j!}. \quad (107)$$

Then $c_{q,0} = 1$, and for every fixed j ,

$$c_{q,j} \longrightarrow \frac{2^j}{j!}. \quad (108)$$

Uniformly for $0 \leq j \leq q$, $0 \leq c_{q,j} \leq 4^j/j!$.

Edge coefficient calculation. For $0 \leq j \leq q$,

$$\beta_{N,p}(\nu+j) = (-1)^q A_{N,p} p^{-j} c_{q,j} + \varepsilon_{q,j}, \quad (109)$$

where

$$\sum_{j=0}^q |\varepsilon_{q,j}| \leq \frac{C_p A_{N,p}}{\nu}. \quad (110)$$

More precisely, with

$$A_+ = \frac{r}{p^{\nu+1}(\nu+1)!}, \quad c_{q,j}^+ = \frac{(q+2)!}{(q+j+2)!} \frac{(2q+j+3)_j}{j!}, \quad (111)$$

one has $\varepsilon_{q,0} = 0$ and

$$\varepsilon_{q,j} = (-1)^{q+1} A_+ p^{-(j-1)} c_{q,j-1}^+, \quad 1 \leq j \leq q. \quad (112)$$

Proof. By [theorem 13](#),

$$\begin{aligned}\log \left[e^{-z} H_{q+1}(z/p)^r H_q(z/p)^{p-r} \right] &= r \sum_{k \geq q+2} \kappa_k^{(q+1)} \frac{(z/p)^k}{(k!)^2} \\ &\quad + (p-r) \sum_{k \geq q+1} \kappa_k^{(q)} \frac{(z/p)^k}{(k!)^2}.\end{aligned} \quad (113)$$

Below degree 2ν , exponentiating does not produce a product of two non-Gaussian coefficients. Substitute [equation \(105\)](#) in the second sum, replace z by $-a$, and compare with [equation \(92\)](#). This gives the first term in [equation \(109\)](#); the factorial quotient is exactly [equation \(107\)](#).

The $q + 1$ blocks start at degree $\nu + 1$. Applying [equation \(105\)](#) with $d = q + 1$ shows that their contribution at degree $\nu + 1 + j$, $0 \leq j \leq q - 1$, is

$$(-1)^{q+1} A_+ p^{-j} c_{q,j}^+.$$

This proves [equation \(112\)](#). No products of non-Gaussian coefficients occur, because the least possible product degree is 2ν . Finally, for $0 \leq j \leq q - 1$, pairing the j numerator and denominator factors in [equation \(111\)](#) gives

$$0 \leq c_{q,j}^+ \leq \frac{4^j}{j!}. \quad (114)$$

Since

$$\frac{A_+}{A_{N,p}} = \frac{r}{(p-r)p(\nu+1)} \leq \frac{C_p}{\nu},$$

we obtain

$$\sum_{j=0}^q |\varepsilon_{q,j}| \leq A_+ \sum_{j \geq 0} \frac{(4/p)^j}{j!} \leq \frac{C_p A_{N,p}}{\nu},$$

which is [equation \(110\)](#). □

Define the one-residue signed comparison function

$$\Lambda_{N,p}(x) = (-1)^q A_{N,p} g(x) \sum_{j=0}^q c_{q,j} p^{-j} L_{\nu+j}(|x|^2). \quad (115)$$

4.2 Fredholm remainder estimates

The Charlier formula identifies which Taylor coefficients belong to the first exterior band. We obtain absolute control of the entire remainder from the determinant, not by taking absolute values inside the Ewens convolution. Specialising the normalised Fredholm identity in [theorem 14](#) to one nonzero pair of times gives

$$H_d(-\rho^2/4) = e^{-\rho^2/4} \det(I - Q_d \mathcal{K}(\rho) Q_d). \quad (116)$$

In particular,

$$\text{Tr}(Q_d \mathcal{K}(\rho) Q_d) = \sum_{i>d} \sum_{j \geq 1} (-1)^{i+j+1} I_{i+j-1}(\rho)^2. \quad (117)$$

Proposition 18 (Fixed-power Fredholm approximation). *Fix $p \geq 1$. For all sufficiently large N , $X_{N,p}$ has a density $f_{N,p}$, and*

$$\|f_{N,p} - g - \Lambda_{N,p}\|_2 \leq C_p A_{N,p} \nu^{-5/4}. \quad (118)$$

Moreover,

$$\|f_{N,p} - g - \Lambda_{N,p}\|_1 = o(A_{N,p} \nu^{-1/4}). \quad (119)$$

Both estimates include the unsmoothed case $p = 1$.

Proof. Put $a = |t|^2/4$ and $\rho = |t|/\sqrt{p}$. We first define the characteristic function, without assuming a density:

$$\varphi_{N,p}(t) := \mathbb{E} \exp\left(it \cdot \frac{\text{Tr}(U_N^p)}{\sqrt{p}}\right) = e^{-a} D_{q+1}(\rho)^r D_q(\rho)^{p-r}. \quad (120)$$

This follows from [equations \(72\) and \(116\)](#) and [theorem 14](#); in particular, we derived its only block reduction from the displacement determinant. Put $\lambda_{N,p} = \mathcal{F}\Lambda_{N,p}$ and

$$\mathcal{E}_{N,p}(t) = \varphi_{N,p}(t) - e^{-a} - \lambda_{N,p}(t). \quad (121)$$

We begin on $0 \leq \rho \leq 2c_0 v$. By [theorem 16](#), with

$$L(\rho) = (p-r)T_q(\rho) + rT_{q+1}(\rho),$$

the finite product satisfies

$$D_q(\rho)^{p-r} D_{q+1}(\rho)^r = 1 - L(\rho) + \mathcal{R}_p^{\det}(\rho), \quad |\mathcal{R}_p^{\det}| \leq C_p(|T_q|^2 + |T_{q+1}|^2). \quad (122)$$

Indeed, insert [equation \(102\)](#), expand the fixed number p of factors, and use [equation \(98\)](#); every term not displayed contains either one E_d or at least two traces.

We next identify and sum the omitted part of L . Grouping [equation \(117\)](#) by the Bessel index gives

$$T_d(\rho) = \sum_{h=1}^{\infty} h(-1)^{d+h} I_{d+h}(\rho)^2. \quad (123)$$

For $x = \rho^2/4$, Vandermonde's identity gives

$$I_n(2\sqrt{x})^2 = \sum_{j=0}^{\infty} \frac{(2n+j+1)_j}{j!(n+j)!^2} x^{n+j}. \quad (124)$$

Indeed, the coefficient of x^{n+j} equals

$$\sum_{v=0}^j \frac{1}{v!(j-v)!(n+v)!(n+j-v)!} = \frac{(2n+j+1)_j}{j!(n+j)!^2}.$$

After the scaling $x = a/p$, the absolute Fourier coefficient of a^{d+h+j} in the h th term is

$$B_{d,h,j}(p) = \frac{h}{p^{d+h+j}} \frac{(2(d+h)+j+1)_j}{j!(d+h+j)!^2}. \quad (125)$$

Equivalently, after Fourier inversion, the absolute Laguerre coefficient from the q blocks is

$$b_{h,j} = (p-r) \frac{h}{p^{q+h+j}} \frac{(2(q+h)+j+1)_j}{j!(q+h+j)!}. \quad (126)$$

Thus $b_{1,0} = A_{N,p}$, and direct division gives

$$\frac{b_{h,j+1}}{b_{h,j}} \leq \frac{4}{p(j+1)}, \quad (127)$$

$$\frac{b_{h,0}}{A_{N,p}} = \frac{h}{p^{h-1}} \frac{v!}{(q+h)!} \leq \frac{h}{(pv)^{h-1}}. \quad (128)$$

Collecting equal powers in [equation \(123\)](#) shows that the part of $-(p-r)T_q$ of degrees $\nu, \dots, 2\nu-1$ is

$$(-1)^q A_{N,p} \sum_{j=0}^q c_{q,j} p^{-j} \frac{a^{\nu+j}}{(\nu+j)!}; \quad (129)$$

this is also the $\ell = 1$ identity in [theorem 17](#). Multiplication by e^{-a} gives exactly $\lambda_{N,p}$.

The raw terms omitted from the q trace satisfy $(h-1) + j \geq \nu$. Writing $u = h-1$ and iterating [equations \(127\) and \(128\)](#), we obtain

$$\sum_{u+j \geq \nu} b_{u+1,j} \leq A_{N,p} \sum_{u+j \geq \nu} \frac{u+1}{(p\nu)^u} \frac{(4/p)^j}{j!} \leq \frac{C_p A_{N,p}}{\nu}. \quad (130)$$

For the last inequality split the sum into $u \geq \nu/2$ and $j \geq \nu/2$: the first part is a geometric tail and the second an exponential-series tail. The $q+1$ blocks begin with

$$A_+ = \frac{r}{p^{\nu+1}(\nu+1)!}, \quad \frac{A_+}{A_{N,p}} = \frac{r}{(p-r)p(\nu+1)} \leq \frac{C_p}{\nu}, \quad (131)$$

and the same ratios show that their complete coefficient mass is $O_p(A_{N,p}/\nu)$.

For every $k \geq 1$,

$$\int_{\mathbb{R}^2} e^{-2a} \left(\frac{a^k}{k!} \right)^2 dt = 2\pi 4^{-k} \binom{2k}{k} \asymp k^{-1/2}. \quad (132)$$

Minkowski's inequality, [equations \(130\) and \(131\)](#) and [equation \(132\)](#) therefore give

$$\|\text{omitted linear trace}\|_2 \leq C_p A_{N,p} \nu^{-5/4}. \quad (133)$$

It remains to control [equation \(122\)](#). Let M_q and M_{q+1} be the positive series obtained from $|T_q(\rho)|$ and $|T_{q+1}(\rho)|$ by replacing every coefficient in [equations \(123\) and \(124\)](#) by its absolute value after the scaling $\rho = |t|/\sqrt{p}$. The ratios [equations \(127\) and \(128\)](#) give

$$\text{mass}(M_q) \leq \frac{A_{N,p}}{p-r} \left(\sum_{u \geq 0} \frac{u+1}{(p\nu)^u} \right) \left(\sum_{j \geq 0} \frac{(4/p)^j}{j!} \right) \leq C_p A_{N,p}, \quad (134)$$

$$\text{mass}(M_{q+1}) \leq \frac{C_p}{p^{\nu+1}(\nu+1)!} \leq \frac{C_p A_{N,p}}{\nu}. \quad (135)$$

Every mode in the first line has degree at least ν , and every mode in the second has degree at least $\nu+1$. Minkowski's inequality and [equation \(132\)](#) therefore yield

$$\|e^{-a} T_q\|_2 \leq \|e^{-a} M_q\|_2 \leq C_p A_{N,p} \nu^{-1/4}, \quad (136)$$

$$\|e^{-a} T_{q+1}\|_2 \leq \|e^{-a} M_{q+1}\|_2 \leq C_p A_{N,p} \nu^{-5/4}. \quad (137)$$

On the central disc, [equation \(98\)](#) also gives

$$\sup_{\rho \leq 2c_0 \nu} \{|T_q(\rho)| + |T_{q+1}(\rho)|\} \leq C/\nu.$$

Consequently, [equation \(122\)](#) gives the complete norm calculation

$$\begin{aligned} \|e^{-a} \mathcal{R}_p^{\det}\|_{L^2(\rho \leq 2c_0 \nu)} &\leq C_p \left\{ \|T_q\|_{L^\infty(\rho \leq 2c_0 \nu)} \|e^{-a} T_q\|_2 + \|T_{q+1}\|_{L^\infty(\rho \leq 2c_0 \nu)} \|e^{-a} T_{q+1}\|_2 \right\} \\ &\leq C_p A_{N,p} \nu^{-5/4}. \end{aligned} \quad (138)$$

This is the required factorial-scale nonlinear estimate; it uses the special signed-trace control in [theorem 16](#), not a generic trace-norm remainder. Combining [equations \(133\) and \(138\)](#) gives

$$\|\mathcal{E}_{N,p}\|_{L^2(\rho \leq 2c_0 v)} \leq C_p A_{N,p} v^{-5/4}. \quad (139)$$

We now treat the whole complementary plane in the single variable ρ . Write $\Psi_d(\rho) = H_d(-\rho^2/4)$. Propositions 2.3 and 2.4 of [\[5\]](#), in the exact form used here, state

$$|\Psi_d(\rho)| \leq \sqrt{2} \exp\left\{-\frac{d^2}{4(2.05 + d^2/\rho^2)}\right\}, \quad \rho \geq 0, \quad (140)$$

$$|\Psi_d(\rho)|^2 \leq \left(\frac{d}{\rho}\right)^d, \quad \rho \geq \frac{7d}{4}. \quad (141)$$

In their notation the corresponding function is denoted by F_d ; it is the characteristic function of $\text{Tr}(U_d)$, their radial variable $|\zeta|$ is our ρ , and the scaling for the original trace is $t = \sqrt{p} \rho$ (or $t = \sqrt{m} \rho$ in the growing power regimes). Thus [equations \(140\) and \(141\)](#) reproduce both the hypotheses and the change of variables used below. Fix $\eta > 0$ and put $\Lambda_q = e^{\eta q}$. From [equation \(140\)](#), for all large q ,

$$\int_{2c_0 v \leq \rho \leq \Lambda_q} |\varphi_{N,p}(t)|^2 dt \leq C_p \Lambda_q^2 e^{-c_p q^2} = o(A_{N,p}^2 v^{-5/2}). \quad (142)$$

For $\rho \geq \Lambda_q$, [equation \(141\)](#) and $N = (p-r)q + r(q+1)$ give

$$|\varphi_{N,p}(t)|^2 \leq \left(\frac{q}{\rho}\right)^{q(p-r)} \left(\frac{q+1}{\rho}\right)^{(q+1)r} \leq \left(\frac{q+1}{\rho}\right)^N, \quad (143)$$

$$\int_{\rho \geq \Lambda_q} |\varphi_{N,p}(t)|^2 dt \leq \frac{2\pi p}{N-2} (q+1)^N \Lambda_q^{2-N} = o(A_{N,p}^2 v^{-5/2}). \quad (144)$$

The Gaussian and the finite polynomial in [equation \(129\)](#) have exponentially smaller complementary tails: here $a = p\rho^2/4 \geq pc_0^2 v^2$, whereas their degrees are at most $2v-1$, so the assertion follows directly from the incomplete Gamma integral. Thus [equations \(139\), \(142\) and \(144\)](#) give the claimed global L^2 estimate.

Finally, the pointwise tail estimate has an integrable square root for all sufficiently large N ; the bounded-frequency part is continuous and bounded. Hence $\varphi_{N,p} \in L^1(\mathbb{R}^2)$ and Fourier inversion proves the existence of a continuous density $f_{N,p}$. Plancherel then identifies the inverse transform of [equation \(121\)](#) with $f_{N,p} - g - \Lambda_{N,p}$ and proves [equation \(118\)](#). We must still convert the L^2 statement to total variation without losing the leading scale. The following truncation makes the argument work for $p=1$ as well.

Write $h_{N,p} = f_{N,p} - g - \Lambda_{N,p}$ and choose

$$R_v = (8 \log(A_{N,p}^{-1}))^{1/2} \asymp (v \log v)^{1/2}. \quad (145)$$

On the disc $|x| \leq R_v$, Cauchy-Schwarz and [equation \(118\)](#) give

$$\int_{|x| \leq R_v} |h_{N,p}(x)| dx \leq \sqrt{\pi} R_v \|h_{N,p}\|_2 \leq C_p A_{N,p} v^{-3/4} \sqrt{\log v} = o(A_{N,p} v^{-1/4}). \quad (146)$$

We now control the three tails separately. Character orthogonality gives

$$\mathbb{E} |\text{Tr}(U_d)|^{2k} \leq k!, \quad H_d(z) \leq e^z \quad (z \geq 0).$$

Applying [equation \(71\)](#) with $z = s^2/(4p)$ yields

$$\mathbb{E} e^{s \operatorname{Re} X_{N,p}} \leq e^{s^2/4}.$$

The same estimate holds in every direction. Chernoff's inequality and four coordinate half-planes consequently give

$$\mathbb{P}(|X_{N,p}| > R) \leq 4e^{-R^2/2}. \quad (147)$$

Also $\mathbb{P}(|Z| > R) = e^{-R^2}$. Finally, Laguerre orthogonality and Cauchy–Schwarz imply

$$\int_{R^2}^{\infty} e^{-u} |L_k(u)| du \leq e^{-R^2/2},$$

while $\sum_{j=0}^q c_{q,j} p^{-j} \leq e^{4/p}$. Hence

$$\int_{|x|>R} |\Lambda_{N,p}(x)| dx \leq C_p A_{N,p} e^{-R^2/2}. \quad (148)$$

At $R = R_v$ all three tails are $O(A_{N,p}^4)$, which is $o(A_{N,p} v^{-1/4})$. Together with [equation \(146\)](#), this proves [equation \(119\)](#). $\square$

4.3 Coherent summation of Laguerre modes

A uniform Laguerre envelope. There is an absolute constant C such that, for every $n \geq 1$,

$$e^{-u/2} |L_n(u)| \leq C n^{-1/4} u^{-1/4}, \quad n^{-1} \leq u \leq n. \quad (149)$$

Proof. Fix a sufficiently large constant ω . For $n^{-1} \leq u \leq \omega$, Szegő's Hilb formula [[25](#), Theorem 8.22.4 and Eq. (8.22.5)], with $\alpha = 0$ and $N = n + 1/2$, gives

$$e^{-u/2} L_n(u) = J_0(2\sqrt{Nu}) + O(n^{-3/4} u^{5/4})$$

uniformly. The fixed-order Bessel estimate $|J_0(v)| \leq C \min(1, v^{-1/2})$ [[9](#), Eq. (10.17.3)] bounds the main term by $C n^{-1/4} u^{-1/4}$. The remainder has the same bound because

$$\frac{n^{-3/4} u^{5/4}}{n^{-1/4} u^{-1/4}} \leq C_\omega n^{-1/2}.$$

For $\omega \leq u \leq n$, write $u = (4n+2) \cos^2 \phi$. Then ϕ is bounded away from zero, while $\pi/2 - \phi \geq c\sqrt{\omega/n}$. Taking ω large enough puts the whole interval inside the uniform range $\varepsilon \leq \phi \leq \pi/2 - \varepsilon n^{-1/2}$ of Szegő's Plancherel–Rotach formula [[25](#), Theorem 8.22.8(a)]. More explicitly, when $\alpha = 0$ that formula has, in magnitude, the form

$$e^{-u/2} |L_n(u)| \leq C (\sin \phi)^{-1/2} n^{-1/4} u^{-1/4} \{1 + O((nu)^{-1/2})\}.$$

Here $u \leq n$ gives $\cos^2 \phi \leq n/(4n+2) < 1/4$, hence $\sin \phi \geq \sqrt{3}/2$, while $u \geq \omega$ gives $\pi/2 - \phi \geq \cos \phi = \sqrt{u/(4n+2)} \geq c_\omega n^{-1/2}$. These are precisely the two endpoint conditions in the cited uniform range. Also $nu \geq n\omega$, so its displayed remainder is uniform and no larger than the leading amplitude. Consequently that amplitude is

$$O((\sin \phi)^{-1/2} n^{-1/4} u^{-1/4}) = O(n^{-1/4} u^{-1/4}),$$

and the two ranges meet at ω , proving [equation \(149\)](#) with one constant. $\square$

The averaging step. Let $0 < \alpha < \beta < \infty$ and $W \in L^1([\alpha, \beta])$. Then, for every fixed $\theta \in \mathbb{R}$,

$$\int_{\alpha}^{\beta} W(v) |\cos(\lambda v + \theta)| dv \longrightarrow \frac{2}{\pi} \int_{\alpha}^{\beta} W(v) dv \quad (\lambda \rightarrow \infty). \quad (150)$$

Proof. The Fourier series of $|\cos x|$ is absolutely convergent, with constant term $2/\pi$ and nonconstant coefficients $O(k^{-2})$. For each partial sum, [equation \(150\)](#) follows from the Riemann–Lebesgue lemma. Uniform convergence of the Fourier series then proves the assertion for every $W \in L^1$. $\square$

Lemma 19 (Laguerre L^1 asymptotics). *As $n \rightarrow \infty$,*

$$n^{1/4} \int_0^{\infty} e^{-u} |L_n(u)| du \longrightarrow C_L := \frac{2^{7/4} \Gamma(3/4)}{\pi^{3/2}}. \quad (151)$$

In addition, there is an absolute constant C such that

$$\int_0^{\infty} e^{-u} |L_n(u)| du \leq C(n+1)^{-1/4}, \quad n \geq 0. \quad (152)$$

For each fixed $j \geq 0$,

$$n^{1/4} \int_0^{\infty} e^{-u} |L_{n+j}(u) - L_n(u)| du \longrightarrow 0. \quad (153)$$

Consequently, for every fixed $p \geq 1$,

$$v^{1/4} \|g \sum_{j=0}^q c_{q,j} p^{-j} L_{v+j}(|\cdot|^2)\|_1 \longrightarrow e^{2/p} C_L. \quad (154)$$

Proof. We use [equation \(149\)](#) for the estimates outside fixed compact subintervals of $(0, \infty)$.

On every fixed compact subinterval of $(0, \infty)$, the same Hilb formula and the Bessel asymptotic give, uniformly,

$$e^{-u} L_n(u) = \frac{n^{-1/4}}{\sqrt{\pi}} e^{-u/2} u^{-1/4} \left[\cos\left(2\sqrt{nu} - \frac{\pi}{4}\right) + o(1) \right].$$

We now split the half-line into five regions. On $0 \leq u \leq n^{-1}$, the defining series gives $|L_n(u)| \leq e^{nu} \leq e$, so the contribution after multiplication by $n^{1/4}$ is $O(n^{-3/4})$. On $[n^{-1}, \delta]$ and $[M, n]$, [equation \(149\)](#) gives

$$n^{1/4} \int_{n^{-1}}^{\delta} e^{-u} |L_n(u)| du \leq C\delta^{3/4}, \quad (155)$$

$$n^{1/4} \int_M^n e^{-u} |L_n(u)| du \leq C \int_M^{\infty} e^{-u/2} u^{-1/4} du. \quad (156)$$

On $[n, \infty)$, Cauchy–Schwarz and Laguerre orthogonality give

$$\int_n^{\infty} e^{-u} |L_n(u)| du \leq \left(\int_n^{\infty} e^{-u} du \right)^{1/2} \left(\int_n^{\infty} e^{-u} L_n(u)^2 du \right)^{1/2} \leq e^{-n/2}. \quad (157)$$

It remains to analyse $[\delta, M]$. Set $u = v^2$. The leading weight becomes $W(v) = 2e^{-v^2/2} v^{1/2}$ and the phase is $2\sqrt{n}v - \pi/4$. Thus the averaging identity [\(150\)](#) gives

$$\begin{aligned} \int_{\delta}^M e^{-u/2} u^{-1/4} \left| \cos\left(2\sqrt{nu} - \frac{\pi}{4}\right) \right| du \\ \longrightarrow \frac{2}{\pi} \int_{\delta}^M e^{-u/2} u^{-1/4} du. \end{aligned}$$

The compact-uniform error tends to zero. First let $n \rightarrow \infty$, then $\delta \downarrow 0$, and finally $M \rightarrow \infty$. The other four regions are uniformly negligible by equations (155) to (157). Consequently,

$$C_L = \frac{2}{\pi^{3/2}} \int_0^\infty e^{-u/2} u^{-1/4} du = \frac{2^{7/4} \Gamma(3/4)}{\pi^{3/2}},$$

This proves equation (151). Without taking limits, the five-region estimates also prove equation (152) for $n \geq 1$; enlarging the constant covers $n = 0$.

For fixed j , on $[\delta, M]$ the amplitudes for $n + j$ and n differ by $o(1)$, while

$$2\sqrt{(n+j)u} - 2\sqrt{nu} = O_j(\sqrt{u/n})$$

uniformly. Hence the compact contribution to $n^{1/4} \int e^{-u} |L_{n+j}(u) - L_n(u)| du$ tends to zero. The same four tail estimates apply to both indices. The short interval between their turning points is exponentially small: for fixed j , Cauchy–Schwarz and Laguerre orthogonality give

$$\int_n^{n+j} e^{-u} |L_{n+j}(u)| du \leq \left(\int_n^{n+j} e^{-u} du \right)^{1/2} \leq e^{-n/2}.$$

Taking first $n \rightarrow \infty$ and then $\delta \downarrow 0$ and $M \rightarrow \infty$ proves equation (153).

For equation (154), first truncate the j -sum. For each fixed truncation, use equations (108) and (153). The estimate

$$c_{q,j} \leq \frac{4^j}{j!}$$

together with equation (152) makes the remaining tail uniformly summable even after multiplication by $v^{1/4}$. Finally, $\sum_{j \geq 0} (2/p)^j / j! = e^{2/p}$. $\square$

Proof of theorem 1. Combine theorems 18 and 19. For every fixed $p \geq 1$, the band summation and the shift estimate first give the sharper Laguerre form

$$\|\Lambda_{N,p}\|_1 = A_{N,p} e^{2/p} \int_0^\infty e^{-u} |L_v(u)| du + o(A_{N,p} v^{-1/4}).$$

Here we used $\int e^{-u} |L_v(u)| du = C_L v^{-1/4} + o(v^{-1/4})$ to replace the asymptotic constant in the band sum by the displayed Laguerre integral. Consequently, first

$$\|f_{N,p} - g\|_1 = A_{N,p} e^{2/p} \int_0^\infty e^{-u} |L_v(u)| du + o(A_{N,p} v^{-1/4}),$$

and then, by equation (151),

$$\|f_{N,p} - g\|_1 = A_{N,p} e^{2/p} C_L v^{-1/4} + o(A_{N,p} v^{-1/4}).$$

Since total variation is half the L^1 distance, this is equations (25) and (26). $\square$

Remark 20 (Check against the known L^2 constant). When $p = 1$, the same first-band calculation and

$$\int_0^\infty e^{-2u} L_n(u)^2 du = 2^{-2n-1} \binom{2n}{n}$$

give

$$\|\mathcal{F}(f_{N,1} - g)\|_2^2 \sim \frac{2e^4 \sqrt{\pi}}{\Gamma(N+2)^2 \sqrt{N}}.$$

This is the exact Fourier- L^2 asymptotic of Courteaut–Johansson–Lambert [5, Theorem 1.2]. Thus the normalisation and the coherent factor e^2 agree with their independently derived constant; the additional step here is the signed comparison-function analysis needed for total variation.

5 Growing powers with fixed block dimension

We prove [theorem 5](#). This fills the regime $m \asymp N$ below the high-power threshold. The proof uses the displacement determinant through [theorem 13](#); it does not invoke the power-map theorem. The Charlier–Ewens formula determines the first non-Gaussian coefficient of each fixed block, while the determinant product controls all subsequent coefficients at once.

For $d \geq 1$ define

$$G_d(z) = e^{-z} H_d(z). \quad (158)$$

The Diaconis–Shahshahani moments give $H_d(z) - e^z = O(z^{d+1})$. At degree $d + 1$, apply [theorem 11](#) with $N = d$ and $\mu = \nu = (1^{d+1})$. Here $D - N = 1$, so only $\mathbf{c} = \mathbf{0}$ occurs in the Charlier kernel and the correction is -1 . Therefore

$$\mathbb{E} |\mathrm{Tr}(U_d)|^{2(d+1)} = (d+1)! - 1,$$

and consequently

$$G_d(z) = 1 - \frac{z^{d+1}}{((d+1)!)^2} + z^{d+2} R_d(z), \quad (159)$$

where R_d is entire. This is the precise point at which the first Charlier exterior band enters the analytic proof.

Let $q, \nu, s, A_{m,r}, B_{m,r}$ be as in [equations \(31\) and \(32\)](#), and put $a = |t|^2/4$. By [equation \(72\)](#),

$$\begin{aligned} \varphi_{N,m}(t) &:= \mathbb{E} \exp \left(it \cdot \frac{\mathrm{Tr}(U_N^m)}{\sqrt{m}} \right) \\ &= e^{-a} G_q(-a/m)^s G_{q+1}(-a/m)^r. \end{aligned} \quad (160)$$

Introduce the two-mode signed comparison function

$$\Lambda_{m,r}^{(2)}(x) = g(x) \left\{ (-1)^q A_{m,r} L_\nu(|x|^2) + (-1)^{q+1} B_{m,r} L_{\nu+1}(|x|^2) \right\}. \quad (161)$$

A differentiated block-product estimate. Fix $\nu \geq 2$ and $K < \infty$. Let $s, r \geq 0$ be integers with $s + r = m$, and suppose that, on $0 \leq a \leq \delta m$,

$$\begin{aligned} u(a) &= \alpha(a/m)^\nu + u_1(a), & |\partial_a^j u_1(a)| &\leq \frac{K(1+a)^{\nu+1+j}}{m^{\nu+1}}, \\ v(a) &= \beta(a/m)^{\nu+1} + v_1(a), & |\partial_a^j v_1(a)| &\leq \frac{K(1+a)^{\nu+2+j}}{m^{\nu+2}}, \end{aligned} \quad 0 \leq j \leq 2,$$

where $|\alpha| + |\beta| \leq K$. If $\delta = \delta(\nu, K) > 0$ is sufficiently small, then, for $0 \leq j \leq 2$,

$$\begin{aligned} &\left| \partial_a^j \left[e^{-a} \left\{ (1+u(a))^s (1+v(a))^r - 1 - s\alpha(a/m)^\nu - r\beta(a/m)^{\nu+1} \right\} \right] \right| \\ &\leq C_{\nu,K} e^{-a/2} (1+a)^{2\nu+2+2j} \left\{ \frac{s}{m^{\nu+1}} + \frac{r}{m^{\nu+2}} + \left(\frac{s}{m^\nu} + \frac{r}{m^{\nu+1}} \right)^2 \right\}. \end{aligned} \quad (162)$$

Proof. The hypotheses and the two retained monomials imply, for $0 \leq h \leq 2$,

$$|\partial_a^h u(a)| \leq \frac{C_{\nu,K}(1+a)^{\nu+h}}{m^\nu}, \quad |\partial_a^h v(a)| \leq \frac{C_{\nu,K}(1+a)^{\nu+1+h}}{m^{\nu+1}}.$$

Moreover, $s|u| + r|v| \leq C_{\nu,K} + C_{\nu,K} \delta^{\nu-1} a$ on the stated interval. To see the constant term explicitly, use $(1+a)^h \leq 2^{h-1}(1+a^h)$: then

$$\frac{s(1+a)^{\nu+1}}{m^{\nu+1}} \leq C_\nu \{1 + \delta^\nu a\}, \quad \frac{r(1+a)^{\nu+2}}{m^{\nu+2}} \leq C_\nu \{1 + \delta^{\nu+1} a\},$$

and the two retained monomials contribute at most $C_{v,K}\delta^{v-1}a$. Decrease δ so that the coefficient of a is at most $1/4$; hence $s|u| + r|v| \leq C_{v,K} + a/4$.

Expand the finite product by the binomial theorem. Its constant and linear parts are $1 + su + rv$; every remaining term is

$$\binom{s}{k} \binom{r}{\ell} u^k v^\ell, \quad k + \ell \geq 2.$$

For $j = 0, 1, 2$, differentiate these monomials by Leibniz's rule and use $\binom{s}{k} \leq s^k/k!$, $\binom{r}{\ell} \leq r^\ell/\ell!$. Summing the resulting exponential series bounds all differentiated nonlinear terms by

$$C_{v,K} e^{a/4} (1+a)^{2v+2+2j} \left(\frac{s}{m^v} + \frac{r}{m^{v+1}} \right)^2.$$

For clarity, this summation can be organised by setting $S_h = s|\partial_a^h u| + r|\partial_a^h v|$. The derivative bounds give $S_h \leq C_{v,K}(1+a)^{v+1+h}(s/m^v + r/m^{v+1})$. The terms of total binomial degree at least two are bounded, for $j = 0, 1, 2$, by Ce^{S_0} times, respectively, S_0^2 , $S_0 S_1$, and $S_1^2 + S_0 S_2$. Since $S_0 \leq C_{v,K} + a/4$, these three bounds imply the preceding display after absorbing $e^{C_{v,K}}$ into the constant. The difference between $su + rv$ and the two displayed monomials in equation (162) is $su_1 + rv_1$; its first two derivatives are bounded by the first two terms in braces in equation (162), with a smaller polynomial factor. Finally, differentiating the common factor e^{-a} does not change its size, and $e^{-a}e^{a/4} \leq e^{-a/2}$. This proves the claim. $\square$

Proposition 21 (Fixed-quotient approximation). *For fixed $q \geq 1$ and all sufficiently large m , the variable $m^{-1/2} \text{Tr}(U_N^m)$ has a density $f_{N,m}$ and, uniformly in $0 \leq r < m$,*

$$\|f_{N,m} - g - \Lambda_{m,r}^{(2)}\|_1 \leq C_q \left\{ \frac{s+1}{m^{v+1}} + \frac{(s+1)^2}{m^{2v}} \right\} + C_q e^{-c_q m}. \quad (163)$$

Proof. We use the weighted Fourier transfer in theorem 15. Expanding equation (159) at $-a/m$ shows that the Fourier transform of $g + \Lambda_{m,r}^{(2)}$ is

$$e^{-a} \left\{ 1 + \frac{(-1)^q s}{m^v (v!)^2} a^v + \frac{(-1)^{q+1} r}{m^{v+1} ((v+1)!)^2} a^{v+1} \right\}. \quad (164)$$

Indeed, equation (88) supplies one factor $1/k!$ and the coefficients in equation (32) supply the other. Thus it remains to estimate the product in equation (160) after its complete linear part has been removed.

We first work in the central region $0 \leq a \leq \delta_q m$, where $\delta_q > 0$ is fixed and sufficiently small. Since the two functions R_q, R_{q+1} in equation (159) and their first two derivatives are bounded on a fixed disc, put $x = a/m$ and write

$$\begin{aligned} G_q(-x) &= 1 + \alpha_q x^v + x^{v+1} R_q(x), & \alpha_q &= \frac{(-1)^q}{(v!)^2}, \\ G_{q+1}(-x) &= 1 + \alpha_{q+1} x^{v+1} + x^{v+2} R_{q+1}(x), & \alpha_{q+1} &= \frac{(-1)^{q+1}}{((v+1)!)^2}. \end{aligned}$$

For $0 \leq j \leq 2$, direct differentiation gives

$$|\partial_a^j \{G_q(-a/m) - 1 - \alpha_q (a/m)^v\}| \leq \frac{C_q (1+a)^{v+1+j}}{m^{v+1}}, \quad (165)$$

$$|\partial_a^j \{G_{q+1}(-a/m) - 1 - \alpha_{q+1} (a/m)^{v+1}\}| \leq \frac{C_q (1+a)^{v+2+j}}{m^{v+2}}. \quad (166)$$

Apply the block-product estimate (162) with

$$u(a) = G_q(-a/m) - 1, \quad v(a) = G_{q+1}(-a/m) - 1, \quad \alpha = \alpha_q, \quad \beta = \alpha_{q+1}.$$

The hypotheses are precisely equations (165) and (166). It follows, for $0 \leq j \leq 2$, that

$$|\partial_a^j \mathcal{E}_{m,r}(a)| \leq C_q e^{-a/2} (1+a)^{2\nu+2+2j} \left\{ \frac{s}{m^{\nu+1}} + \frac{r}{m^{\nu+2}} + \left(\frac{s}{m^\nu} + \frac{r}{m^{\nu+1}} \right)^2 \right\}, \quad 0 \leq j \leq 2, \quad (167)$$

where $\mathcal{E}_{m,r}$ is the difference between equation (160) and equation (164). Thus equation (167) is the complete uniform product estimate, including the endpoint cases $r = 0$ and $s = 1$.

Since polynomial multiples of $e^{-c_q a}$ are square integrable, equations (90) and (167) imply

$$\|\mathcal{E}_{m,r}\|_{L^2(a \leq \delta_q m)} + \|\Delta_t \mathcal{E}_{m,r}\|_{L^2(a \leq \delta_q m)} \leq C_q \left\{ \frac{s}{m^{\nu+1}} + \frac{r}{m^{\nu+2}} + \left(\frac{s}{m^\nu} + \frac{r}{m^{\nu+1}} \right)^2 \right\}. \quad (168)$$

We next verify that extending this estimate to all frequencies costs only $O_q(e^{-c_q m})$. Put

$$\psi_d(u) = H_d(-|u|^2/4), \quad d \in \{q, q+1\}.$$

These are the characteristic functions of $\text{Tr}(U_d)$. On every compact annulus $\varepsilon \leq |u| \leq R$, $|\psi_d(u)| < 1$; equality in the triangle inequality would force $u \cdot \text{Tr}(U_d)$ to be constant modulo 2π , which it is not. Compactness therefore gives a common $\kappa_{q,\varepsilon,R} < 1$. Moreover, the fixed-size Bessel determinant equation (70) and the standard fixed-order Bessel bounds give, for $|\alpha| \leq 4$,

$$|\partial^\alpha \psi_d(u)| \leq C_q (1+|u|)^{-1/2}. \quad (169)$$

Indeed, at $z = -|u|^2/4$ the entries in equation (70) satisfy $I_k(i|u|) = i^k J_k(|u|)$; the phases are removed by diagonal conjugation. Expanding the fixed $d \times d$ determinant, every derivative of order at most four is a finite sum of products of fixed-order Bessel functions and their derivatives. The recurrence relations reduce these derivatives to neighbouring fixed orders, and the standard fixed-order estimate [9, Eq. (10.17.3)] makes each such factor $O_q(|u|^{-1/2})$ for $|u| \geq 1$. The determinant expansion is therefore $O_q(|u|^{-d/2})$, hence $O_q(|u|^{-1/2})$; boundedness on $|u| \leq 1$ follows from analyticity. This proves equation (169). The exponent $1/2$ is sufficient and is sharp for the one-dimensional block.

In equation (160) the two block counts add to m . Apply the compact-annulus bound to every factor. After at most two derivatives, only two factors can be lost, so the resulting bound is $C_q m^2 \kappa_q^{m-2}$. For $|u| \geq R_q$, choose R_q so large that $C_q^2/(1+R_q) < 1$. Keeping the constant contributed by every factor, Leibniz's rule gives, for $|\alpha| \leq 2$,

$$|\partial_u^\alpha \{\psi_q(u)^s \psi_{q+1}(u)^r\}| \leq C_q m^2 C_q^m (1+|u|)^{-m/2}. \quad (170)$$

Consequently,

$$C_q^{2m} \int_{R_q}^\infty u(1+u)^{-m} du \leq \frac{C_q^{2m} (1+R_q)^{2-m}}{m-2} \leq C_q e^{-c_q m}.$$

Changing variables $t = \sqrt{m} u$ introduces only polynomial factors in m ; derivatives in t introduce compensating powers of $m^{-1/2}$. Thus the L^2 norms of the exact characteristic function and its Laplacian on $a > \delta_q m$ are $O_q(e^{-c_q m})$. The Gaussian and the two polynomial comparison modes have the same bound by incomplete Gamma estimates. Hence

$$\|\mathcal{E}_{m,r}\|_2 + \|\Delta_t \mathcal{E}_{m,r}\|_2 \leq C_q \left\{ \frac{s+1}{m^{\nu+1}} + \frac{(s+1)^2}{m^{2\nu}} \right\} + C_q e^{-c_q m}, \quad (171)$$

where we used $r < m$. The same large-frequency estimate makes the characteristic function integrable for large m , and therefore proves existence of $f_{N,m}$.

Finally apply [equation \(89\)](#) to [equation \(171\)](#). Fourier inversion of [equation \(164\)](#) is [equation \(161\)](#); this proves [equation \(163\)](#). $\square$

Proof of [theorem 5](#). Let

$$\eta_{m,r} = \frac{s+1}{m^v}.$$

Because $1 \leq s \leq m$, the right side of [equation \(163\)](#) is $o_q(\eta_{m,r})$: the two algebraic error ratios are at most m^{-1} and $2m^{-q}$, respectively. By polar coordinates,

$$\|\Lambda_{m,r}^{(2)}\|_1 = \int_0^\infty e^{-u} |(-1)^q A_{m,r} L_v(u) + (-1)^{q+1} B_{m,r} L_{v+1}(u)| du.$$

The reverse triangle inequality for norms, followed by [theorem 21](#), now gives [equation \(33\)](#) after division by two.

If $r/m \rightarrow \theta < 1$, then $A_{m,r} \sim (1-\theta)/(m^q \Gamma(q+2))$ and $B_{m,r} = O(m^{-q-1})$, proving [equation \(34\)](#). If $s \rightarrow c$, then

$$m^{q+1} A_{m,r} \rightarrow \frac{c}{\Gamma(q+2)}, \quad m^{q+1} B_{m,r} \rightarrow \frac{1}{\Gamma(q+3)},$$

which gives [equation \(35\)](#). $\square$

Remark 22 (The first strip and the threshold). For $q = 1$, the theorem covers $N/2 < m \leq N$. In the interior of this strip the leading comparison function is $-(2m-N)g_{L_2}/(2m^2)$, reflecting the exact fourth-moment defect

$$\mathbb{E}|\mathrm{Tr}(U_N^m)|^4 = 2m^2 - (2m - N).$$

At $N = m$ one has $r = 0$ and [equation \(33\)](#) reduces to $\frac{1}{4N} \int_0^\infty e^{-u} |L_2(u)| du$, exactly the constant in [theorem 3](#). Thus the below-threshold and above-threshold answers meet without a normalisation gap.

6 Joint growth of power and block dimension

We prove [theorem 4](#). Put $s = m - r$ and

$$A_{N,m} := \frac{s}{m^v \Gamma(v+1)}. \quad (172)$$

The displacement determinant gives, exactly as in [theorem 13](#),

$$\varphi_{N,m}(t) = e^{-a} D_q(\rho)^s D_{q+1}(\rho)^r, \quad a = \frac{|t|^2}{4}, \quad \rho = \frac{|t|}{\sqrt{m}}. \quad (173)$$

This section therefore avoids the power-map theorem.

Let $T_d(\rho) = \mathrm{Tr}(Q_d \mathcal{K}(\rho) Q_d)$ and define the complete first exterior-band signed comparison function by

$$\lambda_{N,m}^{[1]}(t) := -e^{-a} \{s T_q(\rho) + r T_{q+1}(\rho)\}, \quad \Lambda_{N,m}^{[1]} := \mathcal{F}^{-1} \lambda_{N,m}^{[1]}. \quad (174)$$

By [theorem 11](#), this is precisely the $\ell = 1$ Charlier–Ewens term; terms with $\ell \geq 2$ are the nonlinear Fredholm remainder. Finally set the leading one-mode signed comparison function

$$\Lambda_{N,m}^{(0)}(x) := (-1)^q A_{N,m} g(x) L_v(|x|^2). \quad (175)$$

Proposition 23 (Joint-growth approximation). *As $m, q \rightarrow \infty$, uniformly in $0 \leq r < m$,*

$$\|\Lambda_{N,m}^{[1]} - \Lambda_{N,m}^{(0)}\|_1 \leq CA_{N,m} v^{-1/4} \left(\frac{1}{m} + \frac{1}{v} \right). \quad (176)$$

For all sufficiently large m, q , the law of $m^{-1/2} \text{Tr}(U_N^m)$ has a continuous density $f_{N,m}$ and

$$\|f_{N,m} - g - \Lambda_{N,m}^{[1]}\|_1 = o(A_{N,m} v^{-1/4}), \quad (177)$$

uniformly in r .

Proof. Use equations (123) and (124) with $p = m$. After Fourier inversion by theorem 15, the absolute coefficient attached to the q block, displacement $h \geq 1$, and Bessel degree $j \geq 0$ is

$$b_{h,j} = \frac{sh}{m^{q+h+j}} \frac{(2(q+h)+j+1)_j}{j!(q+h+j)!}. \quad (178)$$

In particular, $b_{1,0} = A_{N,m}$. Direct division gives

$$\frac{b_{h,j+1}}{b_{h,j}} = \frac{(2q+2h+2j+2)(2q+2h+2j+1)}{m(j+1)(2q+2h+j+1)(q+h+j+1)} \leq \frac{4}{m(j+1)}, \quad (179)$$

$$\frac{b_{h,0}}{A_{N,m}} = \frac{h}{m^{h-1}} \frac{(q+1)!}{(q+h)!} \leq \frac{h}{(mv)^{h-1}}. \quad (180)$$

The $q+1$ blocks begin with coefficient $r/[m^{\nu+1}\Gamma(\nu+2)]$, and

$$\frac{r}{m^{\nu+1}\Gamma(\nu+2)} \Big/ A_{N,m} = \frac{r}{sm(\nu+1)} \leq \frac{1}{\nu+1}. \quad (181)$$

The remaining coefficients satisfy the same two ratio bounds. Now apply equation (152). Summing equation (179) for $j \geq 1$, then equation (180) for $h \geq 2$, and finally equation (181), proves equation (176). Notice that the estimate is uniform even when $s = m - r = 1$.

We next control all nonlinear exterior terms uniformly. Write $A = A_{N,m}$ and first work with characteristic functions. Set

$$\mathcal{E}_{N,m}(t) = \varphi_{N,m}(t) - e^{-a} - \lambda_{N,m}^{[1]}(t). \quad (182)$$

For $\rho \leq 2c_0 v$, use the logarithmic form in theorem 16:

$$D_q^s D_{q+1}^r = \exp\{-L - R\}, \quad (183)$$

$$L = sT_q + rT_{q+1}, \quad R = s\mathfrak{r}_q + r\mathfrak{r}_{q+1}, \quad (184)$$

$$|R| \leq C\{s|T_q|^2 + r|T_{q+1}|^2\}. \quad (185)$$

This avoids expanding a growing number of determinant factors.

We make the uniformity of the exponent explicit. Put $y = \rho/(2v) \leq c_0$. From equation (97) and Stirling's lower bound,

$$|T_q(\rho)| + |T_{q+1}(\rho)| \leq \frac{C}{v} \{ye^{1+y^2}\}^{2v}. \quad (186)$$

Since $ye^{1+y^2} \leq y/c_0$ on $[0, c_0]$, the right side is at most Cy^2/v . As $a = m\rho^2/4 = mv^2y^2$, this gives

$$|L| \leq \frac{Ca}{v^3}, \quad |R| \leq \frac{C}{v} \{s|T_q| + r|T_{q+1}|\} \leq \frac{Ca}{v^4}. \quad (187)$$

In particular, $|L| + |R| \leq a/4$ for all sufficiently large q , uniformly in m and r .

The sharper factorial form of the same estimate is

$$s|T_q(\rho)| \leq CA \frac{a^\nu}{\nu!} \exp\left(\frac{2a}{m\nu}\right), \quad (188)$$

$$r|T_{q+1}(\rho)| \leq \frac{CA}{\nu} \frac{a^{\nu+1}}{(\nu+1)!} \exp\left(\frac{2a}{m(\nu+1)}\right). \quad (189)$$

For the second line, the ratio of its raw prefactor to the displayed one is $r\nu/[sm(\nu+1)] \leq 1$, so the estimate remains uniform when $s = 1$. Moreover, since s, r are nonnegative integers, [equation \(185\)](#) is bounded by a constant times the square of the sum in [equations \(188\) and \(189\)](#).

Now

$$\mathcal{E}_{N,m}(t) = e^{-a}\{e^{-L-R} - 1 + L\}.$$

Set $z = L + R$. We use the exact decomposition

$$e^{-L-R} - 1 + L = (e^{-z} - 1 + z) - R \quad (190)$$

because the retained linear term is L , not $L + R$. Moreover, since s, r are nonnegative integers,

$$|R| \leq C\{s|T_q|^2 + r|T_{q+1}|^2\} \leq C\{s|T_q| + r|T_{q+1}|\}^2.$$

Now use $|e^{-z} - 1 + z| \leq \frac{1}{2}|z|^2 e^{|z|}$, [equation \(187\)](#), and then [equations \(188\) and \(189\)](#). Since $m\nu \rightarrow \infty$, the harmless exponentials in those two trace estimates are absorbed into $e^{-a/2}$, and we obtain

$$|\mathcal{E}_{N,m}(t)| \leq CA^2 e^{-a/2} \left\{ \frac{a^\nu}{\nu!} + \frac{1}{\nu} \frac{a^{\nu+1}}{(\nu+1)!} \right\}^2. \quad (191)$$

Since $dt = 4\pi da$ for radial integrals, extending the integral to $a = \infty$ and using Gamma integrals gives

$$\begin{aligned} \|\mathcal{E}_{N,m}\|_{L^2(\rho \leq 2c_0\nu)}^2 &\leq CA^4 \int_0^\infty e^{-a} \left\{ \frac{a^\nu}{\nu!} + \frac{1}{\nu} \frac{a^{\nu+1}}{(\nu+1)!} \right\}^4 da \\ &\leq C^{2\nu} A^4, \end{aligned}$$

where the last step follows term by term from $\Gamma(4\nu + j + 1)/(\nu!)^4 \leq C^{2\nu} \nu^{O(1)}$ and Stirling's formula. Absorbing the polynomial factor into the exponential constant yields

$$\|\mathcal{E}_{N,m}\|_{L^2(\rho \leq 2c_0\nu)} \leq C^\nu A^2. \quad (192)$$

We next use ρ consistently on the complementary plane. Let $\Lambda_q = e^{\eta q^2}$, where $\eta > 0$ is a sufficiently small absolute constant. The exact CJL estimates are [equations \(140\) and \(141\)](#). On $2c_0\nu \leq \rho \leq \Lambda_q$ they give

$$|\varphi_{N,m}(t)| \leq 2^{m/2} \exp\left\{-\frac{sq^2 + r(q+1)^2}{18}\right\}.$$

After including the t -area $O(m\Lambda_q^2)$,

$$\int_{2c_0\nu \leq \rho \leq \Lambda_q} |\varphi_{N,m}(t)|^2 dt \leq Ce^{-cmq^2}. \quad (193)$$

For $\rho \geq \Lambda_q$, [equation \(141\)](#) gives

$$|\varphi_{N,m}(t)|^2 \leq \left(\frac{q}{\rho}\right)^{sq} \left(\frac{q+1}{\rho}\right)^{r(q+1)} \leq \left(\frac{q+1}{\rho}\right)^N, \quad (194)$$

$$\int_{\rho \geq \Lambda_q} |\varphi_{N,m}(t)|^2 dt \leq \frac{2\pi m}{N-2} (q+1)^N \Lambda_q^{2-N} \leq C e^{-cmq^2}. \quad (195)$$

We control the linear comparison on the same ρ -scale. The Bessel recurrence $nI_n(\rho) = \frac{\rho}{2}(I_{n-1}(\rho) - I_{n+1}(\rho))$ and $\sum_{n \in \mathbb{Z}} I_n(\rho) = e^\rho$ imply

$$|T_d(\rho)| \leq \sum_{n \geq 1} n I_n(\rho)^2 \leq \rho e^{2\rho}. \quad (196)$$

Thus

$$e^{-a}|L| \leq m\rho \exp(-m\rho^2/4 + 2\rho).$$

Its squared integral from $\rho = 2c_0\nu$ onward is $O(e^{-cmq^2})$, and the Gaussian tail is smaller. Together with [equations \(193\) and \(195\)](#), this proves

$$\|\mathcal{E}_{N,m}\|_{L^2(\rho > 2c_0\nu)} \leq C e^{-cmq^2}. \quad (197)$$

The square root of the power-tail bound is integrable for large N , so $\varphi_{N,m} \in L^1$ and Fourier inversion now proves that the law has a continuous density $f_{N,m}$. Plancherel and [equations \(192\) and \(197\)](#) give

$$\|f_{N,m} - g - \Lambda_{N,m}^{[1]}\|_2 \leq C^\nu A^2 + C e^{-cmq^2}. \quad (198)$$

It remains to compare both errors uniformly with the leading scale. Put

$$R_{N,m} = \sqrt{8 \log(A^{-1})}.$$

Since $s \leq m$, Stirling's formula gives

$$-\log A \leq C\nu(\log m + \log \nu). \quad (199)$$

Moreover,

$$R_{N,m} \nu^{1/4} C^\nu A \longrightarrow 0, \quad (200)$$

$$\frac{R_{N,m} \nu^{1/4} e^{-cmq^2}}{A} \longrightarrow 0. \quad (201)$$

For the second limit, explicitly,

$$\frac{mq^2}{\nu(\log m + \log \nu)} \longrightarrow \infty$$

for arbitrary $m, q \rightarrow \infty$; the first follows from $A \leq m^{1-\nu}/\Gamma(\nu+1)$. Hence [equation \(198\)](#) implies

$$\|f_{N,m} - g - \Lambda_{N,m}^{[1]}\|_2 = o(A\nu^{-1/4} R_{N,m}^{-1}). \quad (202)$$

On $|x| \leq R_{N,m}$ use Cauchy-Schwarz and [equation \(202\)](#). Outside this disc, the moment-generating argument in [equation \(147\)](#) is uniform in m , so the law and the Gaussian have mass $O(e^{-R_{N,m}^2/2})$. Finally, [equations \(179\) and \(180\)](#) show that the total absolute coefficient mass of $\Lambda_{N,m}^{[1]}$ is $O(A)$; Laguerre orthogonality bounds its exterior mass by $O(Ae^{-R_{N,m}^2/2})$. Since $e^{-R_{N,m}^2/2} = A^4$, these tails are negligible and [equation \(177\)](#) follows. $\square$

Proof of theorem 4. By [theorem 23](#),

$$\|f_{N,m} - g\|_1 = A_{N,m} \int_0^\infty e^{-u} |L_\nu(u)| du + o(A_{N,m} \nu^{-1/4}).$$

Divide by two and use [theorem 19](#) to obtain [equations \(29\) and \(30\)](#). $\square$

7 The independent-phase regime $m_N \geq N$

We now prove [theorem 3](#). This regime genuinely has a different probabilistic mechanism, so only the Fourier–Laguerre transfer from [theorem 15](#) is shared with the fixed-power proof.

By Rains’s high-power theorem [22, 23], for every $m \geq N$ the eigenvalues of U_N^m are independent and uniform on the unit circle. Therefore

$$\frac{\text{Tr}(U_N^m)}{\sqrt{N}} \stackrel{d}{=} S_N := \frac{1}{\sqrt{N}} \sum_{j=1}^N \xi_j, \quad \xi_j = e^{i\Theta_j}, \quad (203)$$

where the Θ_j are independent and uniform on $[0, 2\pi]$. Let $J = J_0$. Then

$$\Phi_N(t) := \mathbb{E} e^{it \cdot S_N} = J(|t|/\sqrt{N})^N.$$

Define

$$R_N(t) = J(|t|/\sqrt{N})^N - e^{-|t|^2/4} \left(1 - \frac{|t|^4}{64N} \right). \quad (204)$$

High-frequency remainder.

$$\|R_N\|_2 + \|\Delta R_N\|_2 = O(N^{-2}). \quad (205)$$

Proof. Choose $\delta > 0$ below the first positive zero of J . The expansion

$$\log J(s) = -\frac{s^2}{4} - \frac{s^4}{64} + s^6 \eta(s^2)$$

holds for $|s| \leq \delta$, with η analytic. Put $y = |t|^2$, $T_N = N^{1/8}$,

$$a_N(y) = \frac{y^2}{64N}, \quad b_N(y) = \frac{y^3}{N^2} \eta(y/N), \quad u_N = -a_N + b_N.$$

Using the exact Taylor remainder

$$e^u = 1 + u + u^2 \int_0^1 (1 - \theta) e^{\theta u} d\theta$$

and differentiating this identity gives, for $j = 0, 1, 2$ and $0 \leq y \leq N^{1/4}$,

$$\left| \frac{d^j}{dy^j} \left\{ J(\sqrt{y/N})^N - e^{-y/4} (1 - a_N(y)) \right\} \right| \leq \frac{C}{N^2} P_j(y) e^{-y/8}.$$

The radial identity [equation \(90\)](#), with $a = y/4$, controls $|t| \leq T_N$.

For $T_N \leq |t| \leq \delta \sqrt{N}$, analyticity gives $\log J(s) \leq -cs^2$, $|(\log J)'(s)| \leq Cs$, and $|(\log J)''(s)| \leq C$. Thus the function and its Laplacian are bounded by a fixed polynomial times $e^{-c|t|^2}$, giving $O(e^{-cN^{1/4}})$.

For $s = |t|/\sqrt{N}$, the Bessel equation gives exactly

$$\Delta_t(J(s)^N) = (N-1)J(s)^{N-2}J'(s)^2 - J(s)^N.$$

Choose C_0, s_0 so that $|J(s)| + |J'(s)| \leq C_0 s^{-1/2}$ for $s \geq s_0$, and then choose $M > \max\{\delta, s_0, C_0^2\}$. On $\delta \leq s \leq M$, $\sup |J(s)| = \rho < 1$, so the contribution is polynomial in N times ρ^N . On $s \geq M$,

$$|J(s)^N| + |\Delta_t(J(s)^N)| \leq CN C_0^N s^{-N/2},$$

whose squared integral is bounded by

$$\frac{CN^3M^2}{N-2} \left(\frac{C_0^2}{M} \right)^N.$$

The Gaussian comparison has exponentially small tails. Combining the three regions proves [equation \(205\)](#). $\square$

Proof of [theorem 3](#). The large-frequency estimate used to prove (205) also shows, for all sufficiently large N , that

$$\Phi_N \in L^1(\mathbb{R}^2).$$

Fourier inversion therefore gives a continuous density for S_N ; denote it by f_{S_N} . Let $r_N = \mathcal{F}^{-1}R_N$. By [equations \(89\) and \(205\)](#), $\|r_N\|_1 = O(N^{-2})$. The case $k = 2$ of [equation \(88\)](#) gives

$$f_{S_N}(x) - g(x) = -\frac{1}{2N} L_2(|x|^2) g(x) + r_N(x).$$

Hence

$$d_{\text{TV}}(S_N, Z) = \frac{1}{4N} \int_0^\infty e^{-u} |L_2(u)| du + O(N^{-2}).$$

The roots of $L_2(u) = 1 - 2u + u^2/2$ are $2 \pm \sqrt{2}$, and

$$\int_0^\infty e^{-u} |L_2(u)| du = 4e^{-2} (\sqrt{2} \cosh(\sqrt{2}) - \sinh(\sqrt{2})).$$

Now use [equation \(203\)](#). $\square$

8 Local Poisson approximation for spectra of U^m

We now prove [theorem 6](#). The preceding sections concern the Gaussian approximation of a global Fourier statistic. The local mechanism is different: Rains's decomposition produces independent CUE blocks, and, when their largest intensity tends to zero, a fixed microscopic interval sees every block, up to a smaller-order error, as a rare zero-or-one point source. The first departure from a Poisson process is therefore the second Poisson–Charlier correction to the local count.

Let $\mathcal{N}(B)$ be the space of finite simple counting measures on B , equipped with its counting-measure sigma-field. We use the total-variation distance defined in (17). For a finite signed measure σ , $\|\sigma\|_{\text{var}}$ denotes its variation norm; we do not write d_{TV} when one of the arguments is signed.

8.1 Spectral block decomposition at the local scale

Recall the unfolded process $\Xi_{N,m}$ from (19). Fix $L > 0$ and an interval $B \subset \mathbb{T}_N$ of length L . When $N > L$, rotation invariance makes its location immaterial.

With q, r as in (??), define

$$d_a = \begin{cases} q+1, & 1 \leq a \leq r, \\ q, & r < a \leq m. \end{cases} \quad (206)$$

Blocks of size zero are empty. Rains's power-map theorem [22, 23] identifies [equation \(19\)](#) in law with the superposition of independent unfolded CUE processes X_a of dimensions d_a . Each CUE block is simple

almost surely, so its restriction is an element of $\mathcal{N}(B)$. With respect to Lebesgue measure on $\mathbb{T}_N$, the kernel of the a th block is

$$K_a(x, y) = \frac{1}{N} \sum_{\ell=0}^{d_a-1} e^{2\pi i \ell(x-y)/N}. \quad (207)$$

Its intensity is constant:

$$w_a := K_a(x, x) = \frac{d_a}{N}, \quad \sum_{a=1}^m w_a = 1. \quad (208)$$

Consequently, the quantities in [equation \(36\)](#) can equivalently be written as

$$\gamma_{j;N,m} = \sum_{a=1}^m w_a^j, \quad \beta_{N,m} = \sum_{a=1}^m w_a^2, \quad \varepsilon_{N,m} = \max_a w_a. \quad (209)$$

In particular,

$$\begin{aligned} \beta_{N,m} &= \frac{r(q+1)^2 + (m-r)q^2}{N^2} \\ &= \frac{1}{m} + \frac{r(m-r)}{mN^2}. \end{aligned} \quad (210)$$

The second identity also holds when $m > N$: then $q = 0$, $r = N$, and $\beta_{N,m} = 1/N$.

The power sums in [\(209\)](#) have a direct probabilistic meaning. A window of length L receives mean $L w_a$ from block a . Replacing that block by a Bernoulli source therefore creates a Poisson–binomial count with parameter power sums $L^j \gamma_{j;N,m}$. The quadratic sum $\beta_{N,m}$ measures the chance that two potential points originate in the same source and hence sets the first non-Poisson scale. The cubic sum controls the next term in the convolution logarithm, while γ_4 controls the error made by discarding the spatial repulsion inside one CUE block. Finally, $\varepsilon_{N,m}$ is the largest individual block intensity and is the small parameter needed for uniform Taylor expansion.

This hierarchy also explains the unrestricted condition in the theorem. If $m \leq N$, then $\varepsilon_{N,m} \asymp 1/m$; if $m > N$, it is $1/N$. Thus $N, m \rightarrow \infty$ is exactly what makes every source rare, regardless of the relative rate of the two parameters. Moreover,

$$\gamma_{3;N,m} \leq \varepsilon_{N,m} \beta_{N,m}, \quad \gamma_{4;N,m} \leq \varepsilon_{N,m}^2 \beta_{N,m}, \quad \beta_{N,m} \leq \varepsilon_{N,m}.$$

Hence every remainder on the right of [\(37\)](#) is $o(\beta_{N,m})$. The theorem's leading term is therefore uniform across the transition $m \asymp N$, not merely on fixed-ratio subsequences.

For later use, let $P_2(\cdot; L)$ be the monic Charlier polynomial from [equation \(42\)](#). Explicitly,

$$P_2(k; L) = (k - L)^2 - k, \quad A(L) = \frac{1}{4} \mathbb{E}[P_2(Z_L; L)]. \quad (211)$$

8.2 Poisson–Charlier expansion for sparse block counts

Let π_L denote the law of $Z_L \sim \text{Pois}(L)$ on $\mathbb{Z}_{\geq 0}$, put

$$\mathfrak{d} = \delta_1 - \delta_0,$$

and write $*$ for convolution of signed measures on $\mathbb{Z}$.

Poisson–Charlier transfer. For all $j, k \geq 0$,

$$(\pi_L * \mathfrak{d}^{*j})(k) = \pi_L(k) \frac{P_j(k; L)}{L^j}. \quad (212)$$

Proof. The probability generating function of the left side of equation (212) is

$$e^{L(z-1)}(z-1)^j.$$

On the other hand, multiply equation (42), with $\alpha = L$, by $\pi_L(k)z^k$, sum over k , and compare coefficients of $t^j/j!$. This gives

$$\sum_{k \geq 0} \pi_L(k) P_j(k; L) z^k = L^j (z-1)^j e^{L(z-1)},$$

which proves the identity. $\square$

The next lemma is the discrete-convolution analogue of subtracting the complete linear determinant term in theorems 21 and 23. We include the argument so that the local theorem does not depend on a separate corrected-Poisson approximation result.

Lemma 24 (First Poisson–Charlier correction). *Let $I_1, \dots, I_s$ be independent Bernoulli variables with parameters $p_1, \dots, p_s$. Suppose*

$$\sum_{a=1}^s p_a = L, \quad p_* := \max_a p_a \leq \frac{1}{4},$$

put $S = \sum_a I_a$, and define

$$\lambda_j = \sum_{a=1}^s p_a^j, \quad j \geq 2.$$

Then

$$\left| d_{\text{TV}}(\mathcal{L}(S), \mathcal{L}(Z_L)) - \frac{\lambda_2}{4L^2} \mathbb{E}|P_2(Z_L; L)| \right| \leq C_L(\lambda_3 + \lambda_2^2). \quad (213)$$

Proof. Let $\mathcal{M} = \ell^1(\mathbb{Z})$ be the commutative Banach algebra of finite signed measures, with convolution as multiplication and norm $\|\mu\|_1 = \sum_k |\mu(k)|$. Write

$$\exp_*(\mu) = \sum_{n \geq 0} \frac{\mu^{*n}}{n!}.$$

The Bernoulli law with parameter p_a is $\delta_0 + p_a \mathfrak{d}$. Since $\|p_a \mathfrak{d}\|_1 = 2p_a \leq \frac{1}{2}$, its convolution logarithm is absolutely convergent. Therefore

$$\begin{aligned} \mathcal{L}(S) &= \bigstar_{a=1}^s (\delta_0 + p_a \mathfrak{d}) \\ &= \pi_L * \exp_*(\eta), \quad \eta = \sum_{j \geq 2} \frac{(-1)^{j+1} \lambda_j}{j} \mathfrak{d}^{*j}, \end{aligned} \quad (214)$$

because $\pi_L = \exp_*(L\mathfrak{d})$.

Split

$$\eta_2 = -\frac{\lambda_2}{2} \mathfrak{d}^{*2}, \quad \eta_{\geq 3} = \eta - \eta_2.$$

For $j \geq 3$, $\lambda_j \leq p_*^{j-3} \lambda_3$, while $\|\mathfrak{d}^{*j}\|_1 = 2^j$. Hence

$$\|\eta_{\geq 3}\|_1 \leq \lambda_3 \sum_{j \geq 3} \frac{2^j p_*^{j-3}}{j} \leq C \lambda_3. \quad (215)$$

Also $\|\eta_2\|_1 = 2\lambda_2$. The exact exponential remainder in a Banach algebra gives

$$\|\exp_*(\eta) - \delta_0 - \eta\|_1 \leq \frac{1}{2} \|\eta\|_1^2 e^{\|\eta\|_1}.$$

Since $\lambda_3 \leq p_* \lambda_2$ and $\lambda_2 \leq p_* L$, the exponential factor is bounded by a constant depending only on L . Using [equation \(215\)](#) and contraction of the ℓ^1 norm under convolution with π_L , we obtain

$$\|\mathcal{L}(S) - \pi_L * (\delta_0 + \eta_2)\|_1 \leq C_L(\lambda_3 + \lambda_2^2). \quad (216)$$

By the transfer identity [\(212\)](#),

$$(\pi_L * \eta_2)(k) = -\frac{\lambda_2}{2L^2} \pi_L(k) P_2(k; L). \quad (217)$$

Thus, for the signed measure of total mass one $\sigma_2 = \pi_L * (\delta_0 + \eta_2)$,

$$\frac{1}{2} \|\sigma_2 - \pi_L\|_{\text{var}} = \frac{\lambda_2}{4L^2} \mathbb{E}|P_2(Z_L; L)|. \quad (218)$$

The reverse triangle inequality for the variation norm, together with [equations \(216\) and \(218\)](#), proves [equation \(213\)](#). $\square$

8.3 Configuration-level approximation of CUE blocks

We next show that, in a fixed window, the spatial information inside one Rains block enters below the Poisson–Charlier scale.

Proposition 25 (Sparse-block approximation). *Let X_d be a CUE block of dimension d , unfolded onto $\mathbb{T}_N$ and restricted to B , and put $w = d/N$. Assume $Lw \leq 1$. Let Y_w be empty with probability $1 - Lw$ and otherwise consist of one uniform point of B . Then*

$$d_{\text{TV}}(\mathcal{L}(X_d), \mathcal{L}(Y_w)) \leq \frac{\pi^2 L^4}{18} w^4. \quad (219)$$

Proof. We first record a general point-process observation. Let X be a finite simple point process on B with intensity measure μ , suppose $p = \mu(B) \leq 1$, and put $M = X(B)$. Let Y be empty with probability $1 - p$ and otherwise contain one point with law μ/p , with the statement understood trivially if $p = 0$.

Let Q_n be the restriction of the law of X to the cardinality sector $\{\xi : \xi(B) = n\}$, and let $Q_1^\#$ be the corresponding subprobability measure on B in the one-point sector. The intensity measure decomposes as

$$\mu = Q_1^\# + \zeta, \quad \zeta(A) = \mathbb{E}[X(A) \mathbf{1}_{\{M \geq 2\}}].$$

The singleton-sector measure of Y is μ , whereas that of X is $Q_1^\#$. Moreover,

$$Q_0(\mathcal{N}(B)) - (1 - p) = \mathbb{E}[(M - 1) \mathbf{1}_{\{M \geq 2\}}].$$

Because different cardinality sectors are disjoint, their variation norms add, and hence

$$\begin{aligned} 2d_{\text{TV}}(\mathcal{L}(X), \mathcal{L}(Y)) &= \mathbb{E}[(M-1)\mathbf{1}_{\{M \geq 2\}}] + \zeta(B) + \mathbb{P}(M \geq 2) \\ &= 2\mathbb{E}[M\mathbf{1}_{\{M \geq 2\}}]. \end{aligned} \quad (220)$$

It follows that

$$d_{\text{TV}}(\mathcal{L}(X), \mathcal{L}(Y)) \leq \mathbb{E}[M(M-1)]. \quad (221)$$

For the CUE block, the intensity is $w \, dx$, so the zero-or-one process in this construction is exactly Y_w . The determinantal two-point formula and [equation \(207\)](#) give

$$\mathbb{E}[X_d(B)(X_d(B)-1)] = \int_{B^2} (w^2 - |K_d(x, y)|^2) \, dx \, dy. \quad (222)$$

Writing $t = x - y$ and expanding the squared Dirichlet kernel,

$$\begin{aligned} w^2 - |K_d(x, y)|^2 &= \frac{2}{N^2} \sum_{h=1}^{d-1} (d-h) \left\{ 1 - \cos\left(\frac{2\pi ht}{N}\right) \right\} \\ &\leq \frac{4\pi^2 t^2}{N^4} \sum_{h=1}^{d-1} (d-h)h^2 \\ &= \frac{\pi^2 t^2 d^2(d^2-1)}{3N^4} \leq \frac{\pi^2 t^2}{3} w^4. \end{aligned} \quad (223)$$

Here we identify $B \subset \mathbb{T}_N$ with a real interval lift. This is unambiguous because $L < N$, and all differences $x - y$ in the preceding display use that lift. Finally,

$$\int_{B^2} (x - y)^2 \, dx \, dy = \frac{L^4}{6}.$$

Substitution into [equations \(221\) to \(223\)](#) proves the result. $\square$

For each Rains block, let Y_a be its Bernoulli uniform approximation from [theorem 25](#), and define

$$X = \sum_{a=1}^m X_a = \Xi_{N,m} \upharpoonright_B, \quad Y = \sum_{a=1}^m Y_a, \quad S = Y(B). \quad (224)$$

Independence, the product inequality for total variation, and contraction under superposition yield

$$d_{\text{TV}}(\mathcal{L}(X), \mathcal{L}(Y)) \leq \frac{\pi^2 L^4}{18} \gamma_{4;N,m}. \quad (225)$$

Reduction to the count. With Y and S as in [equation \(224\)](#),

$$d_{\text{TV}}(\mathcal{L}(Y), \mathcal{L}(\Pi_B)) = d_{\text{TV}}(\mathcal{L}(S), \mathcal{L}(Z_L)). \quad (226)$$

Proof. Conditional on $S = k$, the k points of Y are independent and uniform on B . This holds for each fixed set of successful Bernoulli sources and therefore remains true after mixing over that set. Conditional on $Z_L = k$, the Poisson process Π_B has the same law. Since distinct cardinality sectors of $\mathcal{N}(B)$ are disjoint, the variation distance between the two mixtures is exactly the variation distance between their mixing weights. $\square$

8.4 Uniform conclusion across power scales

Proof of theorem 6. For the Bernoulli-source process Y in equation (224), as opposed to the actual DPP count,

$$p_a = L w_a, \quad \sum_a p_a = L, \quad \lambda_2 = L^2 \beta_{N,m}, \quad \lambda_3 = L^3 \gamma_{3;N,m}.$$

The assumption $L \varepsilon_{N,m} \leq \frac{1}{4}$ permits application of theorem 24. Together with the reduction (226) and equation (211), it gives

$$d_{\text{TV}}(\mathcal{L}(Y), \mathcal{L}(\Pi_B)) = A(L) \beta_{N,m} + O_L(\gamma_{3;N,m} + \beta_{N,m}^2). \quad (227)$$

The reverse triangle inequality for a metric, followed by equation (225), now yields

$$|d_{\text{TV}}(\mathcal{L}(X), \mathcal{L}(\Pi_B)) - A(L) \beta_{N,m}| \leq C_L (\gamma_{3;N,m} + \beta_{N,m}^2 + \gamma_{4;N,m}). \quad (228)$$

Let $W = X(B) = \Xi_{N,m}(B)$ be the actual local count. Since the counting map contracts total variation,

$$d_{\text{TV}}(\mathcal{L}(W), \mathcal{L}(S)) \leq d_{\text{TV}}(\mathcal{L}(X), \mathcal{L}(Y)).$$

Another application of the reverse triangle inequality, using theorem 24 and equation (225), gives the corresponding count bound. Adding it to equation (228) proves equation (37).

It remains to verify the uniform asymptotic assertion. From equation (209),

$$\gamma_{3;N,m} \leq \varepsilon_{N,m} \beta_{N,m}, \quad \gamma_{4;N,m} \leq \varepsilon_{N,m}^2 \beta_{N,m}, \quad \beta_{N,m} \leq \varepsilon_{N,m}. \quad (229)$$

Moreover,

$$\varepsilon_{N,m} \leq \frac{1}{m} + \frac{1}{N}. \quad (230)$$

Indeed, the largest nonempty block has size at most $N/m + 1$. Thus the right side of equation (37) is at most $C_L \varepsilon_{N,m} \beta_{N,m}$. If $N, m \rightarrow \infty$, then $\varepsilon_{N,m} \rightarrow 0$, with no condition on m/N . Since $A(L) > 0$, division by $\beta_{N,m}$ proves both asserted asymptotics. $\square$

Remark 26 (The three power ranges). The exact factor equation (210) interpolates automatically between all power scales. If $m = o(N)$ and $m \rightarrow \infty$, then $\beta_{N,m} \sim 1/m$. If $m \asymp N$, the same formula covers every fixed quotient and every strip endpoint. If $m \geq N$, then $\beta_{N,m} = 1/N$ and the spectrum of U^m is already a binomial point process of N independent uniform points; its remaining local discrepancy from Poisson is the fixed-total-number correction. Fixed m is excluded: then the local limit is a superposition of finitely many repulsive CUE bulk processes rather than a Poisson process.

Remark 27 (Meaning of the leading term). The leading correction in equation (37) depends only on the local count, through $P_2(Z_L; L)$. The positional error inside individual CUE blocks is $O_L(\gamma_{4;N,m})$, which is smaller than $\beta_{N,m}$. Thus, to first order, the full local configuration differs from Poisson only because its count is slightly under-dispersed; detailed same-block repulsion appears at lower order.

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