0.1 Proof of Theorem ??

Let $k \geq 1$ and $\alpha > 0$, and let X satisfy $\log X/\log q \to \alpha$ as $q \to \infty$ through the primes. We assume GRH for Dirichlet L-functions modulo q, and assume Conjecture **??** with height parameter $Y(q) = q^{\alpha+\varepsilon}$, for some fixed $\varepsilon > 0$. Choose $\eta > 0$ such that $\alpha(1+\eta) < \alpha + \varepsilon$, and put

$$T = X^{1+\eta}, \qquad \kappa = \frac{1}{\log q}. \tag{1}$$

Then $2T \leq Y(q)$ for all sufficiently large q.

For a Dirichlet character χ mod q, define

$$I_\chi(X) := \sum_{n \leq X} \Lambda^{*k}(n)\chi(n) - \mathbf{1}_{\chi=\chi_0} \operatorname*{Res}_{s=1} \left[\left(-\frac{\zeta_q'}{\zeta_q}(s) \right)^k \frac{X^s}{s} \right]. \tag{2}$$

Since $L(s, \chi_0) = \zeta_q(s)$ and

$$\left(-\frac{L'}{L}(s,\chi) \right)^k = \sum_{n \geq 1} \frac{\Lambda^{*k}(n)\chi(n)}{n^s},$$

orthogonality of characters gives

$$\mathcal{W}_k(X;q) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \left| X^{-1/2} I_\chi(X) \right|^2. \tag{3}$$

We replace $X^{-1/2}I_\chi(X)$ by a truncated Perron integral on $\Re z = \kappa$. Set

$$I_{\chi,T}(X) := \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa+iT} \left(-\frac{L'}{L}\left(\frac{1}{2} + z, \chi\right) \right)^k \frac{X^z}{\frac{1}{2}+z} \, dz. \tag{4}$$

By Perron's formula, followed by the shift from $\Re s = 1 + \kappa$ to $\Re s = 1/2 + \kappa$, GRH gives

$$X^{-1/2} I_\chi(X) = I_{\chi,T}(X) + R_\chi(X;T), \qquad R_\chi(X;T) \ll_{k,\alpha,\eta} X^{-\delta} \tag{5}$$

for some $\delta > 0$, uniformly in χ mod q. Indeed, the only pole crossed for the principal character is the pole at $s = 1$, and its residue is precisely removed in (2). The horizontal sides and the truncation error are bounded by the standard GRH estimate for L'/L to the right of $\Re s = 1/2$.

Squaring (5), averaging over characters, and using Cauchy's inequality gives

$$\mathcal{W}_k(X;q) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} |I_{\chi,T}(X)|^2 + o\left((\log q)^{2k-1} \right). \tag{6}$$

Expanding the square and writing the conjugate integral on the line $\Re z_2 = -\kappa$, we obtain

$$\begin{aligned} \mathcal{W}_k(X;q) = {} & \frac{1}{(2\pi i)^2} \int_{\kappa-iT}^{\kappa+iT} \int_{-\kappa-iT}^{-\kappa+iT} \frac{X^{z_1 - z_2}}{\left(\frac{1}{2}+z_1\right)\left(\frac{1}{2}-z_2\right)} \mathcal{B}_k(z_1, z_2; q) \, dz_2 \, dz_1 \\ & + o\left((\log q)^{2k-1} \right), \end{aligned} \tag{7}$$

where

$$\mathcal{B}_k(z_1, z_2; q) := \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \left(\frac{L'}{L}\left(\frac{1}{2}+z_1, \chi\right) \right)^k \left(\frac{L'}{L}\left(\frac{1}{2}-z_2, \overline{\chi}\right) \right)^k. \tag{8}$$

On these contours, with $z = z_1 - z_2$, we have $\Re z = 2/\log q$, $|\Im z_2| \leq T$, and $|\Im z| \leq 2T$. Thus Conjecture **??** applies throughout the box by (1). For $k = 1, 2$ we choose $0 < \xi < c$, while for

$k \geq 3$ we choose any fixed $0 < \xi < 1$. The integrated contribution of the conjectural error is negligible. Indeed, after the same finite Perron-kernel collapse used below, it is

$$\begin{aligned} &\ll (\log q)^{2k-1+\xi} \int_{-2T}^{2T} \frac{(1+\min(|2\kappa+iy|,1)\log q)^{-\theta_{k,c}}}{1+|y|}\, dy \\ &= o\left((\log q)^{2k-1}\right), \end{aligned} \tag{9}$$

using the definition of $\theta_{k,c}$ and the above choice of ξ.

We now insert the conjectural main term. Changing variables from (z_1, z_2) to (z, z_2), the inner z_2-integral is the same finite Perron kernel as in (**??**), with the shifted linear factors coming from $\left(\frac{1}{2}+z+z_2\right)\left(\frac{1}{2}-z_2\right)$. Hence, uniformly for $\Re z = 2\kappa$ and $|\Im z| \leq 2T$,

$$\frac{1}{2\pi i} \int \frac{dz_2}{\left(\frac{1}{2}+z+z_2\right)\left(\frac{1}{2}-z_2\right)} = \frac{1}{1+z} + O\left(\frac{\log(qT)}{T}\right). \tag{10}$$

The error term in (10) is negligible after insertion into the remaining z-integral. Therefore

$$\mathcal{W}_k(X;q) = \frac{1}{2\pi i} \int_{2\kappa-2iT}^{2\kappa+2iT} \frac{X^z}{1+z} \mathcal{K}_k(z;q)\, dz + o\left((\log q)^{2k-1}\right), \tag{11}$$

where

$$\begin{aligned} \mathcal{K}_k(z;q) := \frac{1}{k!^2(2\pi i)^{2k}} &\int_{\mathcal{C}_u^k} \int_{\mathcal{C}_v^k} q^{\frac{1}{2}\left(\sum_{j=1}^k u_j - \sum_{\ell=1}^k v_\ell\right)} \Delta(u)^2 \Delta(v)^2 \\ &\times \left(\sum_{j=1}^k \frac{1}{u_j} + \sum_{\ell=1}^k \frac{1}{v_\ell - z} - \frac{\log q}{2}\right)^k \left(\sum_{j=1}^k \frac{1}{u_j+z} + \sum_{\ell=1}^k \frac{1}{v_\ell} + \frac{\log q}{2}\right)^k \\ &\times \prod_{j,\ell=1}^k (v_\ell - u_j + z) z^{-k^2} \prod_{j=1}^k u_j^{-k}\, du_j \prod_{\ell=1}^k v_\ell^{-k}\, dv_\ell. \end{aligned} \tag{12}$$

Put $L_q = \log q$, and make the changes of variables

$$z = \frac{w}{L_q}, \qquad u_j = \frac{U_j}{L_q}, \qquad v_\ell = \frac{V_\ell}{L_q}. \tag{13}$$

After the change of variables, rename U_j, V_ℓ as u_j, v_ℓ. Since $\log X/\log q \to \alpha$, we have

$$X^{w/L_q} = e^{\alpha w}(1+o(1)) \tag{14}$$

uniformly on the bounded parts of the w-contour. Also, $(1+w/L_q)^{-1} = 1 + O(L_q^{-1})$, and the contribution of this error is $O(L_q^{2k-2})$ by the same degree bounds used in (**??**). Thus (11) becomes

$$\mathcal{W}_k(X;q) = L_q^{2k-1} \mathfrak{I}_k(\alpha;q) + o\left(L_q^{2k-1}\right), \tag{15}$$

where $\mathfrak{I}_k(\alpha;q)$ is the same scaled contour integral as (**??**), with $X^{1/\alpha}$ replaced by q and with the smoothing factor $\int_0^\infty g(v)\, dv$ omitted.

We now evaluate this contour integral by the previous calculation rather than repeating it. Applying Lemma **??**, then Lemma **??**, gives exactly the decomposition displayed in (**??**), with $\log X^{1/\alpha}$ replaced by $\log q$. The vanishing and degree bounds from (**??**) and (**??**) then allow the same contour shift as in (**??**). Thus only the terms $0 \leq r \leq k$ with $r < \alpha$ contribute at the leading scale; if $\alpha \in \mathbb{Z}$, the block $r = \alpha$ is negligible by the same transition-block argument following (**??**). By (**??**), with the smoothing factor removed, we get

$$\mathfrak{I}_k(\alpha;q) = \delta_k(\alpha) + o(1), \qquad \delta_k(\alpha) := \frac{1}{k!^2} \sum_{\substack{0 \leq r \leq k \\ r < \alpha}} \binom{k}{r}^2 P'_{r,k}(\alpha). \tag{16}$$

This is the same coefficient that appears in the asymptotic for $\widetilde{V}_{g,k}(X, H)$ in (**??**), after removing the factor $\int_0^\infty g(v)\,dv$ and using $\log X^{1/\alpha} = \log q$.

Combining (15) and (16), we obtain

$$\mathcal{W}_k(X; q) = (\log q)^{2k-1}\delta_k(\alpha) + o\left((\log q)^{2k-1}\right). \tag{17}$$

Dividing by $(\log q)^{2k-1}$ and letting $q \to \infty$ through the primes gives

$$\mathcal{Q}_k(\alpha) = \delta_k(\alpha), \tag{18}$$

as required.